MANUAL

OF

MECHANICS.

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MECHANICS.

INTRODUCTION.

MECHANICS is the science which treats of the effects of force, whatever may be the source or origin of that force. We are acquainted with two kinds of effects produced by force, viz., pressure and motion. For our knowledge of the first we are indebted to our muscular sense; the latter is made known to us by two of our senses, sight and touch. Mechanics is, therefore, naturally divided into two branches, which treat respectively of these two effects of force. The first branch of Mechanics, which treats of pressure, is called Statics; and the second branch, which treats of motion, is called Dynamics. We shall proceed, without further explanation or definition, to exhibit the elements of these two departments of Mechanics.

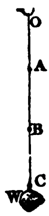
STATICS.

CHAPTER I.

ON FORCES MEETING AT A POINT.

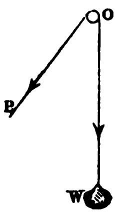
1. Direction of a Force.—2. Magnitude of a Force.—3. Representation of Forces by Lines.—4. Composition of Forces.—5. Duchayla's Proof of the Composition of Forces.—6. The Principle of Moments.

1. **Direction of a Force.**—In estimating the effects of a given force, there are two things to be considered:—first, its direction; and secondly, its magnitude. By the direction of a force, we mean the line in which it tends to produce motion, and the force may be conceived to act at any point of this line: as experience fully shows that the effect of a force, whether in producing pressure or motion, is the same at whatever point in the line of its direction it is applied. If, for example, a weight W be suspended by a string from the point O, it is found that the same force is required to support it, whether it be suspended at the point C, or B, or A.



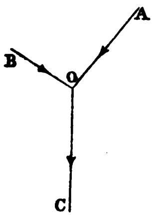
2. **Magnitude of a Force.**—The magnitude of a force is estimated numerically by the weight it will counterpoise, and in order to express this we select some unit of weight, such as a grain, a pound avoirdupois, a ton, &c.; and then denote the magnitude of the force by the number of units and decimal parts of a unit it will sustain. If, for example, a man's hand, acting at P, in

the direction OP, sustain a weight W equal to 1 cwt. 2 qrs. 15 lbs., then the force in the direction OP is said to be equal to 183 lbs., or to 1.63392 cwt., according as the unit selected is a pound or a hundred weight. Having obtained its numerical value, we may then compare it with the forces produced by other causes, by first estimating these in a similar manner.



3. **Representation of Forces by Lines.**—Forces may be represented geometrically by right lines, and this mode of representation has the advantage of exhibiting both direction and magnitude. Having selected a unit of length, such as an inch, a foot, &c., we draw a line in the direction of the given force, and then measure off a segment containing as many units of length as the given force contains units of weight.

Thus, three forces acting at the point O are represented in magnitude and direction by the lines AO, BO, OC, containing, for example, as many feet as there are pounds' weight in the forces represented; while the arrow-heads indicate whether they act from or towards the point O.

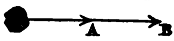


4. **Composition of Forces.**—If two forces act upon the same point of a body, it is always possible to find a single force, whose effect shall be the same as the combined effects of the two given forces. This substitution of one force for two others is called the composition of forces: the force substituted for and equivalent to the other two is called their *Resultant*, and the two forces are called the *Components*.

If the two forces act upon the same point, in the same line, their combined effect will be equal to the sum or difference of the effects of the forces acting separately, according

as they tend to draw the point in the same or in opposite directions.

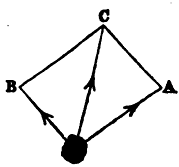
Thus, if OA and OB represent two forces acting upon the point O; in the first case, the effect of the forces will be to draw the body with a force equal to the sum, and in the second case, with a force equal to the difference, of the forces OA and OB.



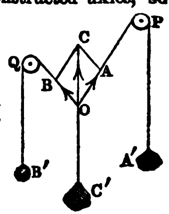
If, however, the forces OA and OB, instead of acting in the same line, act upon the point O in two different directions, then their combined effect is the same as that of a single force OC, whose magnitude and direction are found by the following Proposition:—

PROPOSITION I.—THEOREM.

If two forces, represented in magnitude and direction by the lines OA *and* OB, *act together upon the same point* O, *their combined effect is the same as the effect of a single force* OC, *which is represented in magnitude and direction by the diagonal of the parallelogram formed by drawing* BC *and* AC *parallel to* OA *and* OB.



Let P and Q be two pulleys (with carefully constructed axles, so as to diminish friction as much as possible), fixed in a vertical wall or board, and having their wheels in the same plane parallel to the wall. Let A′ and B′ be two weights attached to silk cords passing over the pulleys, and knotted to each other at the point O; at this point let another weight C′ be attached by means of a silk cord. Let the system be now abandoned to its own action, and it will soon settle into a fixed position, represented in the figure.



Forces equal to the weights A′ and B′ now act at the point O, in the directions OP and OQ, and the effect of these is counteracted by the weight C′, acting vertically downwards. Therefore the weight C′ is equal to, and acts in the opposite direction to, the *Resultant* of A′ and B′. If now the line OA be measured off in the direction OP, containing as many inches as there are ounces in the weight A′; and the line OB be measured off in the direction OQ, containing as many

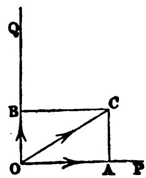
inches as there are ounces in the weight B′, and the parallelogram OACB be completed; then it will be found, on trial, that the vertical line OC′, produced backwards, will always pass through the point C, and that OC contains as many inches as there are ounces in the weight C′. Hence, if two forces, OA and OB, act upon a point O, their combined effect is equal to the effect of a single force represented in magnitude and direction by the diagonal of the parallelogram OACB.—Q. E. D.

The statement of facts contained in this Proposition is usually called the Principle of the Parallelogram of Forces. The forces OA and OB are called Component Forces, and OC is called the Resultant Force.

Care must be taken in constructing the parallelogram of forces, that the components both act *from* the angle of the parallelogram from which the diagonal is drawn.

We are, by means of this law, enabled to replace two forces acting at a point by a single force which will produce the same effect; this substitution is commonly called the Composition of Forces. It is often necessary, for convenience of proof or illustration, to conceive a single force as divided into two forces acting in different directions; this is called the Resolution of Forces, and is effected by the parallelogram of forces in the following manner.

Let OC represent the force which we wish to resolve in the directions OP and OQ; from the point C draw the lines CA and CB parallel to OQ and OP; then it is evident from the parallelogram of forces, that OC is the resultant of the forces OA and OB, and that the effect of the forces OA and OB is the same as the effect of the force OC.



It is found by experiment that *no force can produce any effect in a direction at right angles to itself;* and for this reason, the angle QOP is generally assumed a right angle, because in this case the components OA and OB are totally independent of each other; and each repre-

sents the whole effect of the force OC estimated in its own direction.

EXAMPLES.

1. Two equal forces act on the same point, making with each other an angle of 120°. Find the magnitude and direction of the resultant.

Ans. The resultant bisects the angle, and is equal in magnitude to each of the components.

2. The resultant of two forces acting at right angles is double the lesser of the components. Find the direction of the resultant.

Ans. It divides the angle between the components into two parts, of which one is 60°, and the other 30°.

3. *If two forces act at right angles, the square of the resultant is equal to the sum of the squares of the components.*

For, if OAC (fig. p. 5) be a right-angled triangle, the square of OC will be equal to the sum of the squares of OA and AC, or OB, which is equal to AC.—Q. E. D.

4. Two forces, equal to 30 and 40 lbs. respectively, act at a point at right angles to each other. Find the resultant. *Ans.* 50 lbs.

5. Two forces act at a right angle, one equal to 43 lbs., and the other to 22 lbs. Find the resultant. *Ans.* 48.3 lbs.

6. The resultant of two forces, acting at a right angle, is 56 lbs., and one of the components is 31 lbs. Find the other.

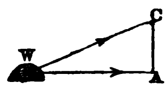
Ans. 46.6 lbs.

7. *The effect of a force estimated in a direction different from its own is found by multiplying the force by the cosine of the angle made by it with the direction in which it is to be estimated.*

For (fig. p. 5) the effect of the force OC in the direction OA is the component OA; but OA is equal to OC multiplied by the cosine of AOC.—Q. E. D.

8. Find the effect of a force of 23 lbs. in a direction making with the line of the force an angle of 32°. On referring to the table of sines and cosines, we find cosine $32° = 0.84805$; multiplying this decimal into 23 lbs., we obtain for the value of the resolved force 19.5 lbs.

9. A weight W is drawn along the ground by a rope OC, which makes with the horizon an angle COA of 12°. If the force OC required to draw the weight be 142 lbs., a smaller force OA acting horizontally will draw it. Find the horizontal force.



Ans. 138.89 lbs.

10. Let the force OA be 120 lbs. Find how much this force should be increased, if the weight be drawn at an angle of 17°.
Ans. 5.48 lbs.

11. The resultant of two forces, acting at right angles, divides the angle between them into 37° and 53°, and is equal to 471 lbs. Find the components. *Ans.* 376.15 lbs. and 283.45 lbs.

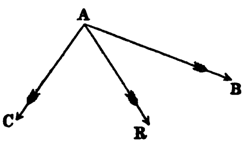
5. **Duchayla's Proof of the Composition of Forces.** —The proof of the composition of forces given in the preceding section, although sufficient for practical purposes, and for establishing the Science of Statics on an experimental basis, is yet wanting in the mathematical precision which is afforded by an *a priori* proof of this fundamental doctrine; we have, therefore, added, in the present edition, to the former proof, the simplest and most elegant of the *a priori* proofs of the composition of forces, which is attributed to Duchayla. It consists of a series of Propositions, which we here reproduce, in order; but the student who is contented to allow his knowledge of mechanics to rest on the basis of experiment, and does not require an *a priori* argument to fortify it, may pass on to section (6).

In Duchayla's proof, the following Postulates are assumed:—

Postulate 1.—*Two forces meeting at a point cannot have more than one resultant.*

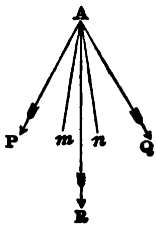
Postulate 2.—*The resultant of two forces meeting at a point, lies in the plane containing the components, and is intermediate in direction.*

Thus, let AB and AC represent two forces acting at the point A, in the senses indicated by the arrows; the direction of their resultant AR is assumed to lie in the plane BAC, intermediate to AB and AC, and in the sense indicated by the arrow.



Proposition A.

The resultant of two equal forces meeting at a point must bisect the angle between them.

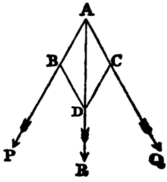


Statement.—Let AP and AQ be two equal forces meeting in a point A; their resultant AR must bisect the angle PAQ.

Proof.—If it be possible, let some other line A*m* be the direction of the resultant, and let A*n* be drawn, making the angle RA*n* equal to the angle RA*m*; then whatever reason can be assigned why A*m* should be the direction of the resultant, a similar reason can be assigned for A*n*; therefore A*m* and A*n* are both the resultant of AP and AQ, which is contrary to the Postulate: in like manner it can be shown that no other line not bisecting the angle PAQ can be the direction of the resultant; therefore the line AR is the direction required.—Q. E. D.

Proposition B.

The resultant of two equal forces meeting at a point is in the direction of the diagonal of the lozenge formed by the forces.



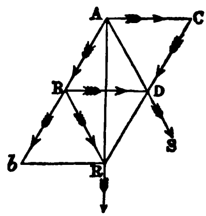
Statement.—Let AP and AQ be two equal forces, represented by the lines AB and AC; and let the lozenge ABDC be completed; the resultant of the forces is in the direction of the diagonal AD.

Proof.—Since AB and AC are equal, and AC and BD also equal (Euc. I. 34), AB is equal to BD; therefore the angles BAD and BDA are equal (Euc. I. 5); but BDA is equal

to CAD (Euc. I. 29); therefore BAD is equal to CAD, and the diagonal AD bisects the angle PAQ; but the resultant AR also bisects the angle PAQ (Prop. A); therefore, &c.—Q. E. D.

Proposition C.

The resultant of two forces meeting at a point, of which one is double the other, is in the direction of the diagonal of the parallelogram formed by these forces.



Statement.—Let AC and Ab be two forces, Ab being double AC; completing the parallelogram ACRb, the resultant of AC and Ab is in the direction of the diagonal AR.

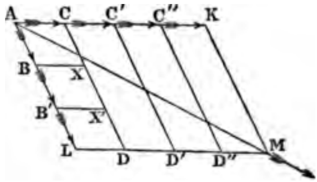
Construction.—Bisect Ab in B, and draw BD parallel to AC and bR, and join AD and BR.

Proof.—The forces AC and Ab are equivalent to AC, AB, and Bb; but AC and AB are equivalent to a resultant in the direction AD (Prop. B), or to CD and BD, by resolving the force AS at the point D; the force CD passes through the point R; and BD and Bb, which remain, being compounded, give a resultant in the direction BR (Prop. B); if, therefore, the point R were held fast, it would destroy the forces CD, BD, and Bb, which are equivalent to the original forces AC and Ab. Therefore R is a point on the resultant, but A is also a point on the resultant; and, therefore, the line AR is the direction of the resultant of AC and Ab.—Q. E. D.

Corollary.—By a similar demonstration it is shown, that if there be any two parallelograms, CB and Db, in which the resultants follow the directions AD and BR, the parallelogram Cb, formed by them, will also possess the property of indicating by its diagonal AR the direction of the resultant of the forces AC and Ab.

Proposition D.

The resultant of any two commensurable forces meeting at a point, is in the direction of the diagonal of the parallelogram formed by the forces.



Statement.—Let AK and AL be two forces meeting at the point A, and commensurable; the direction of their resultant is the line AM.

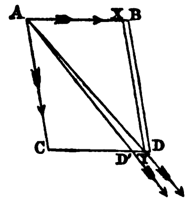
Construction.—Take AC, CC′, C′C″, &c., and AB, BB′, &c., each equal to a common submultiple of AK and AL, and draw CD, C′D′, &c., parallel to AL and MK; also draw BX, B′X′, &c., parallel to AK and ML.

Proof.—Since the forces AC and AB are equal, their resultant is in the direction AX (Prop. B); and since AB and BB′ are equal, the resultant of AC and AB′ is in the direction AX′ (Prop. C); and since B′X′ is equal to B′L, their resultant is in the direction B′D; and, therefore, the resultant of AC and AL is in the direction AD (Prop. C, Corol.). In like manner it can be proved that the resultant of CC′ and CD is in the direction CD′; and therefore the resultant of AC′ and AL is in the direction AD′. In like manner it can be shown that the resultant of AC″ and AL is in the direction AD″, and so on, until, finally, the resultant of AK and AL is proved to be in the direction AM.—Q. E. D.

Proposition E.

If two incommensurable forces meet at a point, their resultant lies in the direction of the diagonal of the parallelogram formed by the forces.

Statement.—Let AB and AC denote two forces, not commensurable, meeting at a point A; their resultant lies in the direction AD.



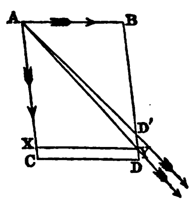
CASE I.

Construction.—If it be possible, let AD′, intermediate between AC and AD, be the direction of the resultant. Let a submultiple of AC be taken, less than DD′, and measured into AB as often as possible to the point X, leaving a remainder XB smaller than DD′, and draw XY parallel to AC and BD.

Proof.—Since the forces AX and AC are commensurable, their resultant lies in the direction AY (Prop. D); and this resultant, compounded with the remainder XB, will give a final resultant of the forces AB and AC, which lies in the angle BAY (Post. 2), and therefore, *a fortiori*, intersects the line CD to the right of the point D′; therefore AD′ is not the direction of the resultant of AB and AC. In like manner it can be shown that no other direction, intermediate between AC and AD, is the direction of the resultant.

CASE II.

Construction.—If it be possible, let some line AD′, intermediate between AB and AD, be the direction of the resultant. Let a submultiple of AB be taken, less than DD′, and measured into AC as often as possible, leaving a remainder XC, less than DD′, and draw XY parallel to AB and CD.



Proof.—Since the forces AB and AX are commensurable, the direction of their resultant is AY (Prop. D); and this resultant, compounded with the remainder XC,

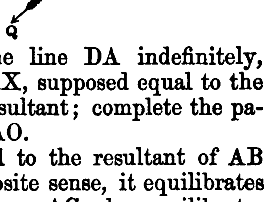
will give a final resultant of the forces AB and AC, which will, *a fortiori*, intersect the line BD below D′; therefore AD′ is not the direction of the resultant of AB and AC. In like manner it can be shown that no other line, intermediate in direction between AB and AD, can be the resultant.

Final Proof.—Since, therefore, no line intermediate between AC and AD is the direction of the resultant; and since no line intermediate between AB and AD is the resultant; and since the resultant of AB and AC must lie between them (Post. 2); therefore the diagonal AD is the direction of the resultant of the two forces.—Q. E. D.

Proposition F.

If two forces meet at a point, the diagonal of the parallelogram formed by them, represents their resultant in magnitude as well as in direction.

Statement.—Let two forces P and Q, represented by AB and AC, meet in the point A; their resultant R is represented in magnitude as well as in direction by the line AD, the diagonal of the parallelogram ABDC.



Construction.—Produce the line DA indefinitely, and measure off on it a line AX, supposed equal to the unknown magnitude of the resultant; complete the parallelogram AXOB, and join AO.

Proof.—Since AX is equal to the resultant of AB and AC, and acts in the opposite sense, it equilibrates them; and therefore one of them, AC, also equilibrates the other, AB, and the force AX; it must, therefore, be equal and opposite to their resultant; but this resultant is in the direction AO (Prop. E), because AXOB was

made a parallelogram; therefore AC and AO are in the same straight line; therefore AOBD is a parallelogram; therefore AD is equal to OB; but AX is also equal to OB (Euc. I. 34); therefore AX and AD are equal; but AX was assumed equal and opposite to the resultant of AB and AC; therefore AD is that resultant. —Q. E. D.

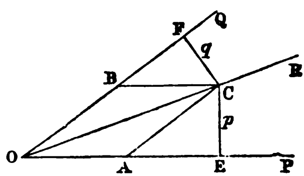
6. **The Principle of Moments.**—DEFINITION.—*The moment of a force with respect to a point is the product of the force into the perpendicular let fall upon its direction from the point.*

What is called by writers on mechanics the Principle of Moments is contained in the two following Propositions, which are converse to each other:—

PROPOSITION II.—THEOREM.

If two forces meet in a point, their moments with respect to any point situated on their resultant are equal and opposite.

Let OP and OQ be the directions of the given forces, and OR the direction of their resultant; from any point C of this resultant let fall perpendiculars CE and CF, upon OP and OQ; and let these perpendiculars be denoted by p, q, and the forces by P, Q; it is required to prove that



$$Pp = Qq.$$

Draw CA and CB parallel to QO and PO; then, since the triangles CEA and CFB are similar,

$$CE : CA :: CF : CB,$$

or, because CA and CB, or BO and OA, are proportional to Q and P,

$$p : Q :: q : P;$$

and, multiplying extremes and means,

$$Pp = Qq.$$

—Q. E. D.

*

PROPOSITION III.—THEOREM.

If the moments of two forces meeting in a point be equal and opposite with respect to any point lying in their plane, that point must be situated on their Resultant.

Using the same construction; since

$$Pp = Qq,$$

we have

$$p : Q :: q : P.$$

But since the triangles CEA and CFB are similar,

$$p : \mathrm{CA} :: q : \mathrm{CB}.$$

And therefore,

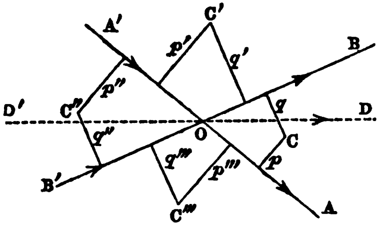
$$Q : P :: \mathrm{CA} : \mathrm{CB}.$$

Therefore the lines OA and OB represent the forces P and Q, and C is a point situated on their Resultant.—Q. E. D.

The physical notion involved in the idea of a moment is that of a *Twist*, which may be right-handed or left-handed; just as the idea involved in a force is that of a *Thrust*, which may be either a push or a pull, according to the direction. The Principle of Moments may therefore be thus stated :—If two forces meeting in a point be applied to a body, every point of the body, situated in the plane of the forces, experiences a *Twist*, which is the algebraic sum of the Twists arising from each force separately, and there exists a line, viz., the Resultant of the forces, along which the Twists are equal and opposite, and equilibrate each other,—and conversely, any point of the body which experiences no *Twist* must lie on the Resultant of the applied forces.

To illustrate this statement, let A′OA be the direction of a force P, and B′OB the direction of a force Q applied to any body; the plane of the forces is divided into two regions by each of the forces.

1st. All points to the right of A′A, as C, C′, will receive a left-handed twist, while all points to the left of A′A, such as C″, C‴, will receive a right-handed twist.



2nd. All points above B′B, as C′, C″, will receive a left-handed twist; and all points below B′B, as C, C‴, will receive a right-handed twist.

Considering the right and left-handed Twists as positive and negative, we have for the points C, C′, C″, C‴ the following Twists :—T, T', T'', T''';

In the angle BOA ; $T = -Pp + Qq$.
In the angle BOA′ ; $T' = -Pp' - Qq'$.
In the angle B′OA′; $T'' = Pp'' - Qq''$.
In the angle B′OA ; $T''' = Pp''' + Qq'''$.

It is evident, therefore, that in the angle AOB′, the Twist is right-handed; in BOA′, left-handed; and in the angles AOB and A′OB′ right-handed or left-handed according as the point selected lies at one side or the other of D′D, the resultant P and Q, along which line the Twist vanishes,—as it is evident it must, because the points situated in this line can only experience a Thrust, and not a Twist.

If D′OD be the line of the resultant, we see, finally, that all points above this line experience a left-handed twist, and all points below this line a right-handed twist. Hence,

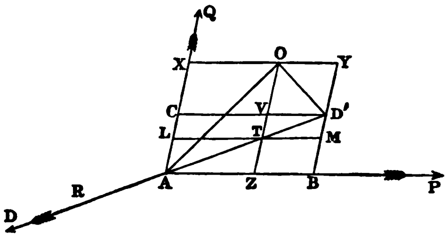
A Moment is the mathematical measure of the amount of a Twist, and is estimated by the product of a weight and a line.

It is easy to show that the moment, or twisting effect of two forces, upon any point, is the same as the moment of their resultant. To prove this, we shall establish the following Proposition:—

PROPOSITION IV.

If three forces, meeting at a point, equilibrate each other, the sum of their moments, with respect to any point, is equal to zero.

Statement.—Let three forces, P, Q, R, represented by the lines AB, AC, AD, meet in the point A, and equilibrate each other; the sum of their moments, with respect to any point O, is equal to zero.



Construction.—Produce the line DA, until AD′ is equal to AD, and complete the parallelogram ABD′C; through the point O, draw XY and OZ parallel to AB and AC; join OA and OD′; and through the point T draw LM parallel to AB.

Proof.—The moment of the force P is left-handed with respect to the point O, and the moments of Q and R are right-handed with respect to the same point; the moment of P, with respect to O, is represented by the parallelogram ABYX; the moment of Q, with respect to O, is represented by the parallelogram ACVZ, which is equal to ABML, because LZ is common to both, and

CT and TB are equal (Euc. I. 43); but the moment of R, with respect to the point O, is equal to double the area of the triangle AOD′, because AD′ was made equal to AD; the triangle AOD′ is equal to OTD′ and OTA together; but OTD′ is half the area of the parallelogram OTMY (Euc. I. 41); and OTA is half the area of OTLX; therefore the triangle AD′O is half the area of the parallelogram LMYX; and therefore the moment of the force R, with respect to the point O, is represented by the parallelogram LMYX; but the parallelogram ABYX is equal to ABML and LMYX together; therefore, the moment of P, with respect to the point O, is equal to the sum of the moments of Q and R, with respect to the same point; but the moment of P is left-handed, and the moments of Q and R are right-handed: therefore, the sum of the moments of P, Q, and R, with respect to the point O, is equal to zero.—Q. E. D.

Corollary.—The moment of the resultant of two forces meeting at a point, with respect to any other point situated in their plane, is equal to the algebraic sum of the moments of the components with respect to the same point.

Exercises on the Composition of Forces.

1. If two forces, P and Q, act upon the same point of a body, and make with each other an angle ϕ, prove that their resultant R is given by the following equation:—

$$R^2 = P^2 + Q^2 + 2PQ\cos\phi.$$

2. If three forces, P, Q, R, equilibrate each other, prove that if $\widehat{QR}$, $\widehat{RP}$, $\widehat{PQ}$, denote the angles between the respective forces,

$$P : Q : R = \sin\widehat{QR} : \sin\widehat{RP} : \sin\widehat{PQ}.$$

3. Two forces, represented by 17 lbs. and 36 lbs., act upon a point, making with each other an angle of 22°; find the magnitude of their resultant. *Ans.* 52.15 lbs.

4. Let the component forces be 26 lbs. and 127 lbs., and the angle between them be 76°. Find the resultant. *Ans.* 135.65 lbs.

5. Two forces, equal respectively to 74 lbs. and 123 lbs., act at an angle of 65°. Find the angles into which their Resultant divides this angle. *Ans.* 23° 30′ and 41° 30′.

6. The angle between two unknown forces is 37°, and their resultant divides this angle into 31° and 6°; find the ratio of the component forces. *Ans.* 0.2029.

7. The Resultant of two forces is 56 lbs., and one of the forces is 22 lbs., and they make an angle of 15°. Find the other component. *Ans.* 35.21 lbs.

8. The resultant of two forces is 10 lbs.; one of them is 8 lbs., and the other is inclined to the resultant at an angle of 36°; find the other force and the angle between them. There are two solutions—

13.51 lbs.; or 2.663 lbs.
132° 43′; 47° 17′.

9. Three forces act perpendicularly to the sides of a triangle at the middle points, and each is proportional to the side on which it acts. Show that they will equilibrate each other.

10. A weight R is sustained as in the figure, page 4, by two weights P and Q, by means of strings passing over two pulleys, not in the same horizontal line. Find the position of rest of the system.

11. Two weights, P and Q, support each other on two inclined planes, making angles with the horizon, a and β respectively, by means of a string passing over the vertex of the planes. Find the ratio of P to Q, and the tension of the string.

Ans. $P : Q :: \sin\beta : \sin a$.
Tension $= P \sin a$.
$= Q \sin\beta$.

12. Three forces equilibrate each other, and are proportional to $\sqrt{3}+1$, $\sqrt{6}$, and 2. Find the angles at which they are inclined to each other. (*Wrigley.*) *Ans.* 105°; 120°; 135°.

13. Four forces, represented by 1 lb., 2 lbs., 3 lbs., 4 lbs., act on a point. The directions of the first and third are at right angles to each other, and so are the directions of the second and fourth; and the angle between the first and second is 60°. Find the magnitude and direction of the Resultant of all the forces. (*Wrigley.*)

Ans. The resultant is 6.88 lbs., and makes an angle of 102° 16′ with the first force.

14. If three forces of 27 lbs., 52 lbs., and 49 lbs., respectively, act upon a point in directions OA, OB, OC, such that AOB is 32°, and BOC is 26°, find their Resultant in magnitude and position.

Ans. The resultant is 119.15 lbs., and makes an angle with OC of 22° 33′.

15. Let the forces be 31 lbs., 16 lbs., 29 lbs., and the angles AOB and BOC equal to 17° and 52°. Find the resultant.

Ans. Resultant is equal to 64.98 lbs., and makes with OC an angle of 39° 45′.

16. If a cord whose length is $2l$ be fastened at A and B, lying in the same horizontal line, at a distance from each other, equal to $2a$; and if a smooth ring upon the cord sustain a weight W; prove that the tension of the cord T is equal to

$$T = \frac{Wl}{2\sqrt{l^2 - a^2}}. \qquad \text{(Newth.)}$$

17. A carriage wheel, whose weight is W and radius r, rests upon a level road; show that the force F necessary to draw the wheel over an obstacle of height h, is

$$F = W\frac{\sqrt{h(2r-h)}}{r-h}. \qquad \text{(Newth.)}$$

This question is solved by equating the moments of the force, and of the weight of the carriage with respect to the summit of the obstacle.

18. If P, Q, R be forces equilibrating at a point, prove that

$$\text{Sin}\ \widehat{PQ} = \frac{\sqrt{(P+Q+R)(Q+R-P)(R+P-Q)(P+Q-R)}}{2PQ}.$$

19. A boat is dragged along a stream, 50 ft. wide, by men on each bank; the length of each rope from its point of attachment to the bank, is 72 ft.; the boat moves straight down the middle of the stream; determine the effective pressure in that direction, if each rope be pulled with a strain of 7 cwts. (*Twisden*). *Ans.* 13.13 cwts.

20. Divide the base, AB, of a triangle ABC, into segments AD, DB, which are to each other in the ratio of n to m, and join CD. Prove that if forces be represented in magnitude and direction by $m \times$ CA and $n \times$ CB, their resultant will be represented in magnitude and direction by $(m + n) \times$ CD.

21. If ABCD be a quadrilateral, and be acted on by forces which are represented in magnitude and direction by AB, AD, CB, CD, show

that the resultant coincides in direction with the line joining the middle points of the diagonals AC, BD, and is represented in magnitude by four times this line.

21. If ABCD be a parallelogram; and forces acting on a point be represented by AB, BD, and CA; investigate the force that would keep the point at rest. *Ans.* + AB.

22. A weight of 5 lbs. is suspended freely from a fixed point by a perfectly flexible string; find what horizontal force applied to the string will draw the upper portion of it 30° out of the perpendicular. *Ans.* 2.886 lbs.

23. Forces proportional to the sides of any plane polygon, acting perpendicularly to these sides at their middle points, will equilibrate.

24. If three forces of 99, 100, and 101 units respectively, act on a point at angles of 120°; find the magnitude of their resultant, and its inclination to the force of 100°. *Ans.* $\sqrt{3}$; 90°.

25. Let forces represented in magnitude and direction by the lines AC, AD, BC, BD act along four edges of the tetrahedron ABCD: prove that their resultant is represented in magnitude and direction by four times the line EF, which joins the middle points of AB and CD, the two remaining and opposite sides of the tetrahedron. (*Artillery Examination.*)

CHAPTER II.

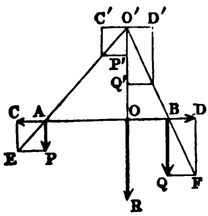
ON PARALLEL FORCES.

1. Composition of parallel forces.—2. Archimedes' proof of the composition of parallel forces.—3. Principle of moments.—4. Centre of parallel forces.—5. Centre of gravity.

1. **Composition of Parallel Forces.**—In the preceding chapter the composition of two forces acting upon the same point of a body has been considered; we shall now consider the case of two forces parallel to each other in direction, and applied at two different points of the body. The composition of such forces may be deduced from the composition of forces meeting at a point, by means of the following Proposition:—

PROPOSITION V.—THEOREM.

If two parallel forces, AP *and* BQ, *act upon the two points* A *and* B, *in the same direction, their combined effect will be equal to the effect of a single force* OR, *equal to their sum, parallel to their direction, and applied at a point* O, *found by dividing the line* AB, *so that* AO *is to* OB *as* BQ *is to* AP, *i. e., inversely as the forces.*



Let two forces, AC and BD, acting in opposite directions, and each equal to the lesser force AP, be introduced into the system. Since these forces would destroy each other if acting alone, they may be introduced without altering the effect of the forces AP and BQ. The forces AP and AC may be compounded (Prop. 1.) into a single force AE, and the forces BQ and BD into the force BF; and these resultant forces themselves may be supposed to act at their point of intersection O′. Let them be resolved at this point into their original components, O′P′ and O′C′, O′Q′ and O′D′. The forces O′C′ and O′D′, being equal and opposite, destroy each other, and the resultant

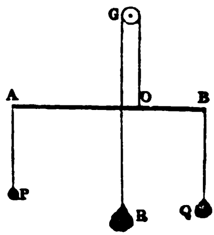
is therefore in the direction O′OR, parallel to the forces AP and BQ, and equal to the sum of OP′ and OQ′, or of AP and BQ.

Also, since the triangles BOO′ and BDF are similar, BO is to OO′ as BD is to DF; but OO′ is equal to OA (because the triangles AOO′ and ACE are similar, and AC was made equal to AP or CE); hence BO is to OA as BD is to DF, or as AP is to BQ.—Q. E. D.

OTHERWISE THUS:

The foregoing proposition has been deduced from the composition of forces meeting at a point, by geometrical reasoning. It might have been experimentally demonstrated in the following manner:—

From the extremities of a light strong rod AB, let two weights, P and Q, be suspended, and let a weight R, equal to their sum, be attached to a silk cord, passing over a pulley G; it will be found on trial that the system will remain at rest if the cord be fastened to the point O, selected so that AO shall be to OB in the ratio of the weight Q to the weight P.



EXAMPLES.

1. Two weights of 126 lbs. and 220 lbs. respectively are suspended from the extremities of a straight bar, 26 inches in length; find the segments into which the resultant divides the bar.

Ans. 16.53 inches, and 9.47 inches.

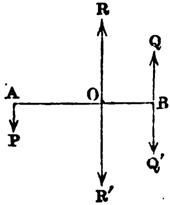
2. Two weights, of 17 stone 6 lbs., and of 2 cwt. 1 qr. 22 lbs., are suspended, as in the last example, to the extremities of a rod 32 feet long; find what point of the rod should be supported in order to sustain the weight.

Ans. The segments are 15.073 feet, and 16.927 feet.

In the case just discussed the parallel forces have the same direction, and their Resultant is always the sum of the forces; but if it be required to find the resultant of two parallel forces, acting in opposite directions, it may be done by the aid of the following Proposition:—

PROPOSITION VI.—PROBLEM.

To find the resultant in magnitude and position of two parallel forces, AP *and* OR, *acting in opposite directions.*



Let the force R be greater than P; find a force Q equal to the difference between R and P; and produce the line AO to B, so that

$$AO : OB :: Q : P,$$

and at the point B apply the force Q, in the direction BQ of the greater force R; the force BQ is the required resultant.

Assuming BQ′ equal and opposite to BQ; by the preceding proposition, the resultant of AP and BQ′ is a force OR′ equal to $P + Q$; but, by construction, the force OR is equal to $P + Q$, and therefore OR is equal to OR′, and since it acts in the same line, and in an opposite direction, it equilibrates OR′, and consequently the force BQ′ equilibrates AP and OR; and therefore its equal and opposite BQ is the resultant of AP and OR.

The resultant therefore of two parallel forces acting in opposite directions is equal to their difference. And, since

$$AO : OB :: R - P : P.$$

Componendo (Euc. v. 18),

$$AB : OB :: R : P.$$

And therefore, the point of application of the resultant of two opposite and parallel forces is found *by producing the line joining their points of application, so that the whole produced line shall be to the produced segment inversely as the forces.* Q. E. I.

In order to find the resultant of two parallel forces, it is necessary, as is shown by the Propositions v. and vi., to cut the line joining their points of application in the inverse ratio of the forces; this section being internal in the case of parallel forces acting in the same direction, and external in the case of opposite parallel forces. It is always possible to perform this section in the first case, but in the latter case, if the opposite forces be equal, we are required to produce a line so that the whole produced line shall be equal to the produced part,

which is impossible. In this exceptional case, therefore, the forces have no resultant, and the system becomes, in fact, what we have called a Twist in the preceding chapter. Such a combination of two forces is frequently called a Pair or a Couple.

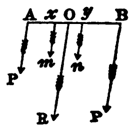
2. Archimedes' Proof of the Composition of Parallel Forces.—Archimedes founded the science of Statics upon a demonstration of the composition of parallel forces, of which the following proof gives the essential parts:—

Postulate 1. Two parallel forces cannot have more than one resultant.

Postulate 2. The resultant of two concurrent parallel forces lies in their plane, and is intermediate between them.

PROPOSITION A.

The resultant of two equal concurrent parallel forces bisects the line joining their points of application.



Statement.—Let A and B be the points of application of two equal parallel forces P and P: their resultant OR must bisect the line AB.

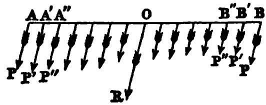
Construction.—If it be possible, let xm be the resultant, and not OR; draw yn, making Ox equal to Oy, and parallel to the forces P.

Proof.—Whatever reason can be assigned why xm should be the resultant of the forces P, a similar reason can be assigned in favour of the resultant yn, which is symmetrically situated; therefore xm and yn are both resultants of the forces P, which is contrary to the Postulate; therefore xm is not the resultant of the forces P; in like manner, it can be shown that no other line, not bisecting the line AB, is the direction of the resultant; therefore, the resultant of the forces P bisects the line AB.—Q. E. D.

Corollary.—The resultant is equal to twice P in magnitude, as the forces all have the same direction.

PROPOSITION B.

The resultant of any number of equal, parallel, concurrent forces, whose points of application are situated at equal distances along the same right line, passes through the middle point of that right line.



Statement.—Let P, P′, P″, &c., be parallel, equal, concurrent forces applied at equal distances along the line AB; their resultant passes through O, the point of bisection of the right line AB.

Proof.—The number of forces is either even or odd.

If it be even, the forces may be combined in pairs, PP, P′P′, P″P″, &c., applied at the extremities of right lines AB, A′B′, A″B″, &c., whose common point of bisection is O; but each of these pairs of forces has a resultant passing through O (Prop. A); therefore the common resultant of all the forces passes through the point O.

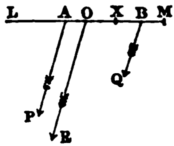
If the number of forces be odd, they may be combined, as before, in pairs, PP, P′P′, P″P″, &c., applied at the extremities of lines AB, A′B′, &c., whose common centre is O, together with a single force applied at the point O; since the resultants of the pairs of forces (Prop. A) and the single odd force all pass through O; therefore the common resultant of all the forces passes through O.

In every case, therefore, the resultant of the parallel, equal, concurrent, and equidistant forces passes through the point of bisection of the line of their application.

Corollary.—The Resultant is equal in magnitude to the sum of all the components.—Q. E. D.

Proposition C.

The resultant of two parallel, concurrent, commensurable forces, divides the line joining their points of application inversely in the ratio of the forces.



Statement.—Let A and B be the points of application of two commensurable, concurrent, parallel forces P and Q, and let the line AB be divided in O, so that AO : OB :: Q : P; the point O is the point of application of R, the resultant of P and Q.

Construction.—Divide the line AB in X, so that AX : XB :: P : Q; and produce it both ways, making AL equal to AX, and BM equal to BX.

Proof.—Let P and Q be to each other in the ratio of the integer numbers m and n; and let P be divided into m equal parts, and Q into n parts equal to the others. Let LX be divided into m, and XM into n equal parts. Since the line LX contains m equal parts, there will be $(m - 1)$ points of section, together with the extremities L and X; let $(m - 1)$ of the submultiples of P be applied at the points of section, and the m^{th} submultiple be divided into two equal parts, and applied at L and X; there will thus be applied at equal distances along the line LX, whose point of bisection is A, $(m - 1)$ equal forces $\left(\frac{P}{m}\right)$, and two equal forces $\left(\frac{P}{2m}\right)$, at the points L and X, and the force P is equivalent to this system of forces (Prop. B, Corol., and Prop. A). In like manner the force Q may be replaced by $(n - 1)$ equal forces $\left(\frac{Q}{n}\right)$, applied at equal distances along a line whose point of bisection is B, together with two equal forces $\left(\frac{Q}{2n}\right)$, applied at X and M: therefore the two forces are

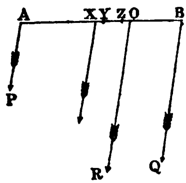
equivalent to $(m + n - 1)$ equal forces applied to the points of section of the line LM, divided into $(m + n)$ equal parts, together with two equal forces $\left(\frac{P}{2m} \text{ and } \frac{Q}{2n}\right)$ applied at its extremities L and M; therefore the Resultant of P and Q is applied at the point of bisection of the line LM (Props. A and B); but AB is equal to half LM, and LO is equal to AB, because LA is equal to AX, equal to OB, and AO is common to both; therefore O is the point of bisection of the line LM; and is, therefore, the point of application of the resultant of P and Q; therefore the point of application of the resultant of two parallel, commensurable, concurrent forces divides the line joining their points of application inversely in the ratio of the forces.—Q. E. D.

Corollary.—The resultant is equal to $P + Q$ in magnitude.

Proposition D.

The Resultant of two parallel, concurrent, incommensurable forces, divides the line joining their points of application inversely as the forces.

Statement.—Let P and Q be two parallel, concurrent, incommensurable forces applied at the points A and B; their resultant will divide the line AB in the point O, so that $P : Q :: \text{OB} : \text{OA}$.



Construction.—If it be possible, let the point of application of the resultant be any point X, different from O. Take a submultiple $\left(\frac{1}{m^{th}}\right)$ of OB, less than XO, and divide it into the line AO, as often as possible (n), leaving a remainder YO; less than XO; and divide the line YO in Z, so that

$$\text{YZ} : \text{ZO} :: n : m.$$

Proof.—Since

$$AY : OB :: n : m,$$

and

$$YZ : ZO :: n : m,$$

$$AZ : ZB :: n : m \text{ (Euc. V. 12)}:$$

take now an equi-submultiple of $P\left(\frac{1}{m^{th}}\right)$, and divide it as often as possible (n times) into Q, leaving a remainder, and let Q' denote the difference between Q and this remainder; then

$$P : Q' :: m : n;$$

and, therefore,

$$P : Q' :: ZB : ZA;$$

therefore Z is the point of application of the resultant of the two commensurate forces P and Q' (Prop. C); and this point of application lies to the right of the supposed resultant X; but since Q is greater than Q' by the above-mentioned remainder, therefore the true resultant of P and Q lies between Z and B (Post. 2), and, *a fortiori*, between X and B; therefore no point X, lying between A and O, can be the point of application of the resultant; and in like manner it can be shown that no point lying between B and O can be the point of application; but the resultant must lie between A and B (Post. 2); therefore O is the point of application of the resultant of P and Q. —Q. E. D.

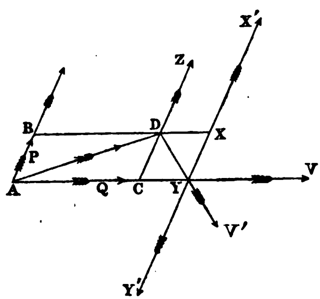
Corollary.—The Resultant of the forces P and Q is equal to their sum $P + Q$.

In order to complete the theory of the Composition of Forces, when the composition of parallel forces is made its basis, it is necessary to add a demonstration to show how the composition of two forces meeting at a point may be deduced from the composition of parallel forces; this Proposition is analogous to that given in the commencement of this chapter, by which we passed from

the composition of forces meeting at a point to that of parallel forces.

PROPOSITION VII.—THEOREM.

If two forces meet at a point, their resultant will coincide in direction with the diagonal of the parallelogram formed by lines representing the forces in magnitude and direction.



Construction. — Let AB and AC represent two forces P and Q; complete the parallelogram ACDB, and produce BD and AC to X and Y, making DX and CY equal to AB; complete the rhombus DCYX by drawing XY.

Proof.—Let two forces XX′ and YY′, each equal to AC or Q, and opposite in direction, be introduced into the system; this will not disturb the forces P and Q already existing, as the forces introduced balance each other. The force Q may be supposed to act along the line YV at the point Y; this force, compounded with the equal force Q along the line YY′, will give a resultant bisecting the angle VYY′ (Prop. A, Chap. I.); this resultant DYV′ must pass through the point D, because CDXY is a rhombus. There remain the forces P, applied as A or B, and Q, parallel to P, applied at the point X; the resultant of these forces, by Archimedes' proof, pass through D, because BD : DX : : Q : P; therefore the entire system of forces may be replaced by two forces DV′ and DZ, both passing through the same point D; therefore D is a point on the resultant of P and Q: but A is also a point on this resultant, therefore

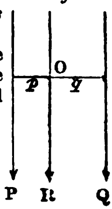
AD, the diagonal of the parallelogram ACDB, is the direction of the resultant of the forces P and Q, represented by AB and AC.—Q. E. D.

It is, therefore, evident that the whole science of Statics may be founded either on the composition of forces meeting in a point, or on the composition of parallel forces, and that either of these compositions being assumed, or proved, the other follows from it by strict geometrical reasoning. We have given both methods of laying the foundation of Statics on *à priori* reasoning, and also the experimental method, which, to some minds, will prove as satisfactory as either.

3. **Principle of Moments.**—The principle of moments explained in Chap. I, with respect to forces meeting in a point, is equally true of parallel forces, as may readily be seen from the following Propositions:—

PROPOSITION VIII.—THEOREM.

If any point O be taken on the resultant of two parallel forces P and Q, the moments of these forces with respect to this point are equal and opposite.



Through the point O draw a line perpendicular to the forces P, Q, and their resultant R, and let p, q be the intercepts on this perpendicular between the point O and the forces P and Q. By Prop. V. it appears that

$$P : Q :: q : p,$$

and therefore

$$Pp = Qq;$$

but Pp is the moment of P with respect to the point O; and Qq is the moment of Q with respect to the point O. Also, since the twist of P upon the point O is left-handed, and that of Q right-handed, these twists, or moments, are opposite—Q. E. D.

PROPOSITION IX.—THEOREM.

If the moments of two parallel forces P and Q, with respect to any point O, be equal and opposite; this point is situated on the resultant.

Using the same construction, since the moments of the forces with respect to O are equal, we have

$$Pp = Qq;$$

and therefore

$$P : Q :: q : p;$$

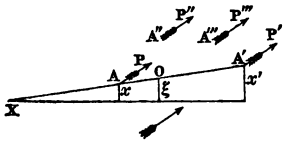
or the point O is situated on the resultant of P and Q, because it divides the line joining the points of application of the forces inversely as the forces.—Q. E. D.

The following theorem respecting the moments of any system of parallel forces is extremely useful in practice:—

PROPOSITION X.—THEOREM.

The moment of the resultant of any number of parallel forces, with respect to any plane, is equal to the sum of the moments of the component forces with respect to that plane.

DEFINITION: The moment of a force with respect to a plane is the product of the force and of the perpendicular let fall on the plane from the point of application of the force.



Let A and A′ be the points of application of two of the forces P, P′; and let O be the point of application of their resultant. Let the line AA′ be drawn intersecting the plane in X, and let perpendiculars x, ξ, x' be let fall on the plane from A, O, and A′. By similar triangles, x, ξ, x' are proportional to XA, XO, XA′, and therefore their differences are also proportional; but, by the principle of moments,

$$P \times \mathrm{AO} = P' \times \mathrm{A'O};$$

or,

$$\mathrm{P}(\xi - x) = \mathrm{P}'(x' - \xi);$$

and

$$(P + P')\xi = Px + P'x'.$$

If the point O and A″ be joined, and the joining line produced to intersect the plane, a similar proof will apply to the parallel forces P″ and (P + P′), whose points of application are A″ and O; calling *X* the perpendicular let fall on the plane from the point of application of their resultant, we find

$$[P'' + (P + P')]X = P''x'' + (P + P')\xi;$$

and, substituting for $(P + P')\xi$, its value already found, and arranging, we obtain

$$(P + P' + P'')X = Px + P'x' + P''x''.$$

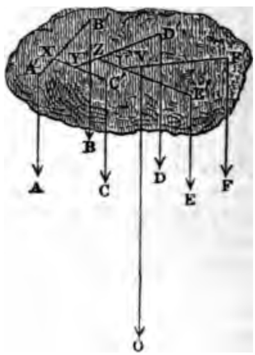
The same proof will apply to any number of parallel forces, and therefore, finally,

$$(P + P' + P'' + \&c.)X = Px + P'x' + P''x'' + \&c. \qquad (1)$$

Q. E. D.

4. **Centre of Parallel Forces.**—*To find the resultant of any number of parallel forces acting upon a body.*

Let A′A, B′B, C′C, &c., be the given forces. First join A′ and B′, and cut the joining line at X inversely in the ratio of the forces A′A and B′B; the resultant of these forces, which is equal to their sum, passes through this point. Next cut the line XC′ in the inverse ratio of the sum of AA′ and BB′ to the force CC′; let Y be the point of section; the resultant of the first three forces AA′, BB′, CC′, which is equal to their sum, passes through this point: next join YD′ and cut it at Z, in the



inverse ratio of the sum of AA′, BB′, CC′, to the force DD′; the resultant of the first four forces AA′, BB′, CC′, DD′, which is equal to their sum, passes through this point, and so on. The point V, which is found last, is the point of application of the resultant of all the forces; and the magnitude of the resultant VO is the sum of all the forces AA′, BB′, &c.

The point V *which has just been found is called the centre of the system of parallel forces*, for as it was found without any reference to the direction of the forces AA′, &c., it is independent of this direction, and is, consequently, a fixed point in the body, through which the resultant of the parallel forces must pass, in whatever direction they may tend.

The position of the centre of parallel forces is found analytically by means of Prop. x. Let P, P', P'', &c., be the given forces, and let x, x', x'', &c., y, y', y'', &c., z, z', z'', &c., be perpendiculars let fall from their points of application on any three planes drawn at right angles to each other; we have the following equations:—

$$\begin{aligned}(\mathrm{P} + \mathrm{P}' + \mathrm{P}'' + \&c.)\xi &= \mathrm{P}x + \mathrm{P}'x' + \mathrm{P}''x'' + \&c.\\ (\mathrm{P} + \mathrm{P}' + \mathrm{P}'' + \&c.)\eta &= \mathrm{P}y + \mathrm{P}'y' + \mathrm{P}''y'' + \&c. \qquad (2)\\ (\mathrm{P} + \mathrm{P}' + \mathrm{P}'' + \&c.)\zeta &= \mathrm{P}z + \mathrm{P}'z' + \mathrm{P}''z'' + \&c.\end{aligned}$$

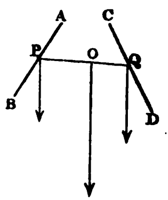
In these three equations, the forces P, P′, &c., are known, and their points of application (x, y, z), (x', y', z'), &c., are also known, the unknown quantities are ξ, η, ζ, the co-ordinates of the point of application of the resultant, which, is therefore, determined.

5. **Centre of Gravity.**—Gravity is the name given to the force of attraction, exerted by the earth on all bodies situated on its surface: the effects of this force are twofold—statical and dynamical, in virtue of which all bodies have weight, and, if unsupported, would fall to the ground. The force of gravity, and consequently the weight of bodies, varies from place to place on the

surface of the earth; but at each place it is a force exerted upon all bodies and parts of bodies, in lines perpendicular to the horizon, and, consequently, parallel. The direction of the force may be found by means of a plumb-line, or by a sheet of water, since the force of gravity can be proved to be perpendicular to the surface of a fluid at rest.

The weight of a body may be considered as the resultant of the weights of all the atoms or indivisible particles of which it is composed; and since the force of gravity, at the same place on the earth's surface, acts in parallel directions, it follows from what has been proved, respecting a system of parallel forces, that the resultant of the weights of all the parts of a body passes through a fixed point, which is the centre of this system of parallel forces. This point is called the *centre of gravity*. The method of finding the centre of a system of parallel forces having been described in the preceding paragraph, it is not necessary here to repeat it, as it is plain that the problem of finding the centre of gravity of a body is the same as that of finding the centre of a system of parallel forces.

Exercises on Parallel Forces.



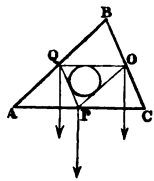
1. To find the common centre of gravity of two homogeneous lines. Let AB and CD be the given lines; since they are homogeneous, the centre of gravity of each line shall be its point of bisection P and Q; join these two points, and cut the joining line inversely as the weights of AB and CD, i. e. so that OP will be to OQ as the weight of CD is to the weight of AB. The truth of this construction is evident, from the consideration that there are two forces parallel to each other, and equal to the weights of AB and CD, applied at the points P and Q, and by Prop. IV. the point O will be the point of application of their resultant.

2. Let the weights of AB and CD, respectively, be 23 lbs. and 42 lbs., and the length of PQ 14 inches. Find the line OQ to three places of decimals. *Ans*. 4.953 in.

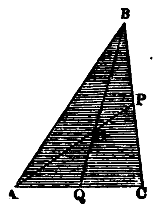
3. Let the weights be 3 cwts. 2 qrs. 15 lbs., and 1 cwt. 3 qrs. 25 lbs., and the line PQ equal 3 feet 7 inches. Calculate the line OP to three places of decimals of feet. *Ans.* 1.261 ft.

4. To find the common centre of gravity of three bodies. Join the centres of gravity of the first and second bodies, and cut the joining line inversely as their weights. Join this point with the centre of gravity of the third body, and cut the joining line inversely in the ratio of the sum of the weights of the first and second bodies to the weight of the third. The point thus found is the centre of gravity required.

5. To find the centre of gravity of the perimeter of a triangle. Let ABC be the given triangle. Bisect each of the sides in O, P, Q; the weights of the three sides respectively may be conceived as concentrated in these three points. The question is thus reduced to Example (4). It is not difficult to prove that the centre of gravity is the centre of the circle inscribed in the triangle OPQ.



6. To find the centre of gravity of a homogeneous thin plate, cut into the form of a triangle. Let ABC be the triangle, which may be conceived as divided into thin bars parallel to AC. The line BQ, joining the angle B with the point of bisection Q of the opposite side, will pass through the centres of gravity, or points of bisection of all these bars. Consequently, the centre of gravity of the triangle lies on this line BQ. In a similar manner it may be shown to lie on the line AP, which is drawn bisecting the side BC. Consequently, it must be at the intersection O of these lines.

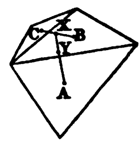


7. *If a body be suspended from a fixed point* O, *it will not be in equilibrium unless the line* OG, *joining the point of suspension with the centre of gravity, be vertical.*

For, let the line OG not be vertical, and let GC represent the weight of the body acting vertically downwards; let this force be resolved into two forces, GA and GB (Prop. 1); it is evident that the force GA will be destroyed by the reaction of the fixed point O, while the force GB will cause the body to move towards the vertical line OV. Hence the body cannot be at rest unless the line GO be vertical.

8. Hence may be deduced an easy mode of finding, experimentally, the centre of gravity of a plane figure of irregular shape. Let it be suspended to a fixed point, by means of a cord attached to any point of its surface; when it has arrived at a state of rest, the line of the cord will be vertical, and will also pass through the centre of gravity. By marking on the body the line of the cord, and producing it downwards, we shall obtain one line passing through the centre of gravity. Repeating the experiment, with a different point of suspension, we shall obtain a second line containing the centre of gravity: consequently, the intersection of these two lines will be the centre of gravity of the body.

9. To find the centre of gravity of a polygon. Let the polygon be divided into triangles by diagonals drawn from any angle, and let the centre of gravity of each of these triangles be found (Ex. 6); let A, B, C be the centres of gravity; join BC, and cut the joining line in X inversely as the weights of the first two triangles; join XA, and cut it in Y inversely as the sum of the weights of the first two triangles to the weight of the third triangle. The point Y is the centre of gravity of the polygon.



10. If a system of n forces acting on a point be represented in magnitude and direction by right lines drawn from that point; prove that their resultant will be represented by n times the right line drawn from the point to the centre of gravity of equal masses placed at the extremities of the right lines representing the components.

11. Prove that the centre of gravity of a tetrahedron lies at the common intersection of the four lines drawn from each angle to the centre of gravity of the opposite face; and that each of these lines is quadrisected at the centre of gravity.

12. If a and b denote the parallel faces of a trapezium, and h the perpendicular distance between them; prove by Prop. x. that if X and Y denote the distance of its centre of gravity from a and b respectively;

$$X = \frac{h}{3} \times \frac{2b + a}{a + b},$$

$$Y = \frac{h}{3} \times \frac{2a + b}{a + b}.$$

13. The centre of gravity of a trapezium lies on the line joining the points of bisection of its parallel sides, and divides that line in the ratio $2a + b : 2b + a$.

This famous theorem was first proved by Archimedes in the xvth Prop. of his First Book on Equilibrium—Πέρι ἐπιπέδων ἰσοῤῥοπικῶν, ἢ κεντραβαρῶν.

14. Let a and b denote the diameters of the non-concentric spheres forming a shell; and let D denote the distance between their centres; find the distance from the centre of the sphere a of the centre of gravity of the shell.

$$x = \frac{b^3 D}{a^3 - b^3}.$$

15. A bar of iron 15 inches long, weighing 12 lbs., and of uniform thickness, has a weight of 10 lbs. suspended from one extremity; where must a fulcrum be placed that it may just balance upon it?

Ans. $4\frac{1}{11}$ in. from the weight.

16. A straight bar is used in the experimental proof of the composition of parallel forces, its length is l, and its weight W; if any two weights P and Q be suspended from its extremities; find the position of the fulcrum which will verify the law of composition of parallel forces.

Ans. $x = \frac{l}{2} \times \frac{2Q + W}{P + Q + W}$.

17. If a tetrahedron M, M′, M″, M‴, be formed by joining the centres of gravity of four masses m, m', m'', m'''; prove that these masses are proportional respectively to the tetrahedrons, whose common vertex is the centre of gravity of the four masses, and whose bases are the respectively opposite faces of the tetrahedron M, M′, M″, M‴.

18. Seven bodies of equal weight are placed so that their centres of gravity coincide with as many angles of a cube, whose diagonal is a; find the distance of their common centre of gravity from the unoccupied angle of the cube.

Ans. $\frac{4a}{7}$.

19. A bar of uniform thickness weighs 10 lbs., and is 5 ft. long; weights of 9 lbs. and 5 lbs. are suspended from its extremities; on what point will it balance?

Ans. The centre of the bar is 5 in. distant from the fulcrum.

20. A beam 30 ft. long balances itself on a point at one-third of its length from the thicker end; but when a weight of 10 lbs. is suspended from the smaller end, the prop must be moved two feet towards it, in order to maintain the equilibrium. Find the weight of the beam.

Ans. 90 lbs.

21. A uniform bar, 4 ft. long, weighs 10 lbs., and weights of 30 lbs.

and 40 lbs. are appended to its two extremities; where must the fulcrum be placed to produce equilibrium?

Ans. The fulcrum is 3 in. distant from the centre of the bar.

22. A bar of iron, of uniform thickness, 10 ft. long, and weighing $1\frac{1}{2}$ cwt., is supported at its extremities in a horizontal position, and carries a weight of 4 cwt. suspended from a point distant 3 ft. from one extremity. Find the pressures on the points of support.

Ans. 3.55 cwt.
1.95 „

23. A triangular slab of uniform thickness is supported at its three angular points Whatever be the form of the triangle, the pressures on the props are all equal.

24. A bar, each foot in length of which weighs 7 lbs., rests upon a fulcrum distant 3 ft. from one extremity, what must be its length, that a weight of $71\frac{1}{2}$ lbs. suspended from that extremity may just be balanced by 20 lbs. suspended from the other. *Ans.* 9 ft.

25. What is the height from the ground of the centre of gravity of a pyramid formed of four eight-inch shells touching each other?

Ans. $4 + 2\sqrt{\frac{2}{3}} = 5.632$ in.

26. Five equal parallel forces act at five of the angles of a hexagon, whose diagonal is a; find the point of application of their resultant.

Ans. On the diagonal passing through the sixth angle, at a distance from it of $\frac{3}{5}a$.

27. The sides of a triangle are 3, 4, 5; and its inscribed circle has been removed; find the centre of gravity of the remainder.

Ans. Its distances from the sides 3 and 4, are $\dfrac{8-\pi}{5}$ and $\dfrac{6-\pi}{5}$ respectively.

28. A heavy triangle ABC is suspended successively from the angles A and B, and the two positions of any side are found to be at right angles to each other. Prove that

$$5c^2 = a^2 + b^2.$$

29. A body P suspended from one end of a lever without weight is balanced by a weight of 1 lb. at the other end of the lever; when the fulcrum is removed through half the length of the lever, it requires 10 lbs. to balance P; determine the weight of P.

Ans. 5 lbs. or 2 lbs.

30. A line is drawn cutting off one quarter of a square, whose

side is a, find the distance of the centre of gravity of the remainder of the square from the centre of gravity of the whole square, (1) when the line is parallel to a side of the square, (2) when it is parallel to a diagonal of the square.

Ans. $\frac{a}{8}$, and $a \times \frac{3\sqrt{2} - 2}{18}$.

31. On a uniform straight bar weighing 5 lbs., and 5 ft. long, weights of 1, 2, 3, 4 lbs. are hung at the distances 1, 2, 3, 4 ft. respectively from the extremity. Find the distance from the centre of the bar, of the fulcrum, on which the whole will rest. *Ans.* 4 in.

32. The greatest angle of a given isosceles triangle is 120°; from the vertex and from the middle points of the equal sides perpendiculars are drawn on the base; supposing the perimeter of the triangle and the perpendiculars to be uniform rods of the same material, find the distance of the centre of gravity of this framework from the vertex, the perpendicular from vertex being a.

Ans. $a \times \frac{13 + 8\sqrt{3}}{24 + 8\sqrt{3}}$.

CHAPTER III.

ON MACHINES IN EQUILIBRIUM.

1. Principle on which Equilibrium of Machines is determined.—2. The Lever.—3. The Wheel and Axle.—4. The Inclined Plane.—5. The Moveable Inclined Plane.—6. The Screw.—7. The Pulley.

1. **Principle on which Equilibrium of Machines is determined.**—A machine is an instrument by means of which pressure or motion may be transmitted from one point to another, and altered both in magnitude and direction. In all machines there are two forces to be considered: first, the Power applied to the machine; and, secondly, the Resistance, to overcome which the Power is used. Two forces will not equilibrate each other unless they be equal and opposite, from which it might be supposed that the Power and Resistance should be equal to each other. This would necessarily be the case if the machine were free, i. e. not in contact with fixed points or surfaces. But as all machines contain fixed points or fulcra, and sometimes fixed lines or surfaces, and as these are capable of resisting pressure, and so supporting part of the Resistance, it follows that the Power may be less than the Resistance, and bear any ratio to it, according to the position of the fixed points or surfaces. It is to be observed, that while fixed points are capable of resisting pressure in all directions, fixed lines and surfaces can only do so in a direction at right angles to themselves. The principle, according to which the equilibrium of machines is to be found, is contained in the following Proposition:—

PROPOSITION XI.—THEOREM.

Let the Power and Resistance be considered as two forces applied to the machine, and let the direction of their resultant be determined;

the machine will be in equilibrium when this resultant passes through a fixed point, or is perpendicular to a fixed line or surface in the system.

The truth of this Proposition is evident from the consideration, that if the resultant do not pass through a fixed point, &c., it will cause motion, and consequently the machine cannot be in equilibrium.

COROLLARY.—The strain or pressure sustained by the fixed point or surface is the resultant of the Power and Resistance.

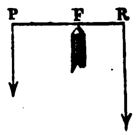
There are some machines of simple construction, commonly called the Mechanical Powers, to one or other of which, or to combinations of which, every machine, however complicated, may be ultimately reduced. They may be classified as follows:—

I. The Lever, including	1. The Lever proper. 2. The Wheel and Axle.
II. The Inclined Plane,	3. The Inclined Plane. 4. The Wedge. 5. The Screw.
III. The Cord, ...	6. The Pulley.

They are six in number, reducible to three distinct machines, involving distinct principles.

It would be impossible to discuss all the combinations of these powers that occur in practice; we shall, therefore, devote a section to each power, and add some examples which will show how the combinations are to be discussed.

2. The Lever.—A Lever is a rigid bar, capable of motion about a fixed point or axis, which is called the fulcrum. There are three kinds of levers.



In the first kind, the fulcrum F is situated between the points P and R, at which the Power and Resistance are applied.

In the second kind, the Resistance is applied at a point situated between the fulcrum and the point at which the power is applied.

In the third kind, the point at which the power is applied is situated between the fulcrum and the point at which the resistance is applied.

In each of these cases there is a fixed point (viz., the fulcrum F) in the system; consequently, by Prop. XI., the resultant of the Power and Resistance must pass through this point, which therefore supports a strain whose magnitude is measured by this resultant.

If the Power and Resistance be parallel in direction, and be applied perpendicularly to a straight lever,

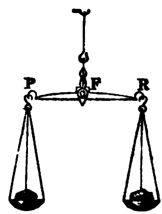
The Power is to the Resistance inversely as the arms of the lever.

For, by Prop. XI., the resultant of the forces P and R passes through the point F, and therefore, by Prop. II.,

$$P : R :: \mathrm{FR} : \mathrm{FP}.$$

—Q. E. D.

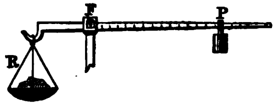
If the arms of the lever be equal, then it is requisite for equilibrium that the Power and Resistance shall be also equal.



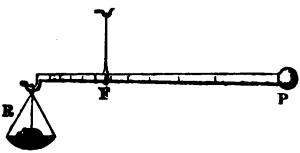
This case occurs in the common balance, in which the Power and Resistance are the weights in each pan of the scales; if the balance be true, the arms are equal, and therefore the weights are also equal; but if the arms be unequal, the weights will be also unequal in the inverse ratio. The simplest mode of detecting a false balance is to transpose the weights; for they cannot counterbalance each other, when so transposed, unless the arms and weights be both equal.

Besides the balance with equal arms, there are other balances in common use for ascertaining the weights of bodies for commercial purposes. The *Roman Steelyard* is one of these. It consists of a beam, or bar of iron, resting upon a pivot F, and having one arm longer than the other. To the extremity of the shorter arm is attached

a scale-pan, and the position of the pivot is so adjusted that the weight of the scale-pan and shorter arm balances the weight of the longer arm; a weight or counterpoise P can be moved along the longer arm of the balance, while the commodity to be weighed is placed in the scale-pan. When the counterpoise balances the contents of the scale-pan, it is evident, from the principle of the lever, that their weights will be as RF to FP; but the first and third terms of this proportion are constant, viz., the counterpoise P, and the distance RF; therefore, the second and fourth terms of the proportion bear a constant ratio to each other; i. e. the commodity to be weighed is proportional to the line FP. By dividing the latter into equal parts, we can therefore determine the weight of a given article.



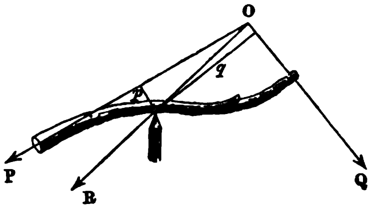
Another balance of this kind in common use is the *Danish Balance*, which differs from the *Roman Steelyard* in having its fulcrum F moveable, and the counterpoise P fixed at one extremity; to the other extremity is attached a scale-pan. In order to graduate the beam, we place successively 1, 2, 3, &c. lbs. in the scale-pan, and mark the point on the beam at which the fulcrum must be held so that the beam shall be horizontal; at these points we engrave the figures 1, 2, 3, &c.; the intervals between these figures must be divided by adding ounces to the corresponding pounds weight in the scale-pan, and so dividing the intervals experimentally. The intervals between the figures 1, 2, 3, &c., will not be equal, as in the case of the *Roman Steelyard*.



If the lever be not straight, or the power and resistance *P* and *Q* be not applied parallel to each other in

direction, the rule given for finding the equilibrium in page 42 cannot be applied.

Let the accompanying figure represent the general case. Produce the directions of the power and resistance to meet in the point O, and from O draw OR through the fulcrum; it follows, from Prop. XI., that this line is the direction of the resultant of P and Q, and therefore, since the fulcrum is a point on the resultant, if p and q be the perpendiculars drawn from the fulcrum to the lines OP and OQ; by Prop. II.,



$$Pp = Qq,$$

or,

$$P : Q :: q : p.$$

Or,

The Power is to the Resistance inversely as the perpendiculars let fall on their directions from the fulcrum.

The principles just laid down enable us to solve the following questions:—

1st. Given the magnitude and direction of the Resistance applied to one extremity of a lever; to determine the Power which should be applied in a given direction to the opposite extremity so as just to sustain it.

2nd. Knowing in magnitude and direction the Power and Resistance applied to the arms of a lever; to calculate in magnitude and direction the strain on the fulcrum.

EXAMPLES.

1. Let the resistance applied perpendicularly to one extremity of a lever be 247 lbs.; the lever being 22 in. long, and having its fulcrum 3 in. distant from the extremity to which the resistance is applied.

Find the amount of the force which should be applied to the other extremity of the lever, at an angle of 27°, so as to balance the given resistance. *Ans.* 85.9 lbs.

2. Let the Resistance be caused by a nail fastened in a board, and be equal to 224 lbs.; let the lever used to draw it be a hammer, which measures 2 in. from the point where the claw grasps the nail's head to the fulcrum, and let the handle be 13 in. long. Find the force to be applied perpendicularly to the extremity of the handle, which will exactly counterbalance the resistance. *Ans.* 34.46 lbs.

3. In a lever of the First Order, let the Power be 217 lbs., the Resistance 725 lbs., and the angle between them 126°. Find the strain on the fulcrum. *Ans.* 622.7 lbs.

4. In a lever of the Second Order, let the Power and Resistance be 310 lbs. and 422 lbs., and the angle between them 37°; find the strain on the fulcrum. *Ans.* 255.4 lbs.

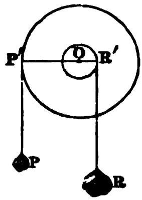
5. In a lever of the Third Order, let the Power and Resistance be 217 lbs. and 100 lbs.; and the angle between them 42°; find the strain on the fulcrum. *Ans.* 157.6 lbs.

6. If the Power and Resistance in a straight lever of the First Order be 17 lbs. and 32 lbs., and make with each other an angle of 79°; find the strain on the fulcrum. *Ans.* 39 lbs.

3. **The Wheel and Axle.**—The wheel and axle is one of the simplest and most useful of the modifications of the lever: it consists, as its name expresses, of a large wheel, or simply spokes fastened perpendicularly on an axle, which is supported by two rings or sockets, in which it revolves, which act as fulcra, and must sustain the pressure caused by the resultant of the Power and Resistance. The Power is applied to the circumference of the wheel; and the Resistance to the circumference of the axle; in directions at right angles to their radii.

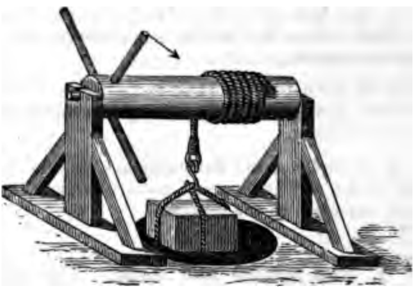
The annexed figure represents a cross section of the wheel and axle, in which OP′ is the radius of the wheel, OR′ the radius of the axle, and P and R the Power and Resistance, both acting vertically downwards. It is evident, from inspection of the figure, that P′R′ may be considered as a lever of the first order, hav-

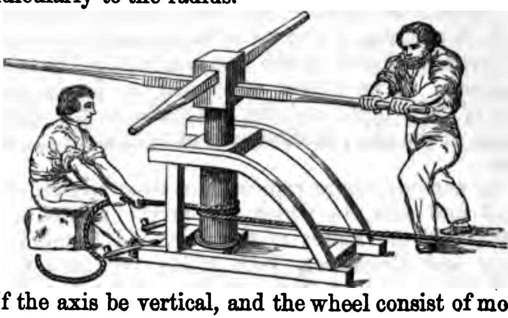
ing its fulcrum at O, and the weights P and R applied at its extremities; hence, the condition requisite for equilibrium in the wheel and axle may be expressed as follows:—



There will be equilibrium in the wheel and axle when the Power is to the Resistance as the radius of the axle is to the radius of the wheel.

One of the most common examples of the wheel and axle is the *windlass*, in which the Power is applied by means of a winch handle or spokes. If the labourer who turns the handle exert his force always at right angles to it, the conditions of equilibrium in the windlass will be precisely the same as in the wheel and axle; for the winch handle describes a circle which corresponds with the wheel, provided the force act perpendicularly to the radius.

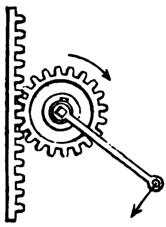




If the axis be vertical, and the wheel consist of move-

able spokes against which a pressure is exerted by means of men pushing with their shoulders, the instrument is called a *capstan*, and the conditions of equilibrium are the same as in the wheel and axle. This instrument is used on board ship for raising the anchor.

If a small wheel with cogs, or teeth, be placed upon the axle, and this be made to work into a vertical bar fitted with cogs, this combination is also an instance of the wheel and axle; it is called the *rack and pinion*, and is evidently the same as a windlass, in which the Resistance is applied by means of cogged wheels instead of by means of a rope or chain; the mechanical effect of both instruments is the same.



EXAMPLES.

1. What force will be required to work the handle of a windlass, the Resistance to be overcome being 1156 lbs., the radius of the axle being 6 inches, and of the handle 2 feet 8 inches?

Ans. 216.75 lbs.

2. A weight of 17 lbs. just balances a weight of 79 lbs. on a wheel and axle. What will be the radius of the axle, if that of the wheel be 17 inches? *Ans.* 3.65 inches.

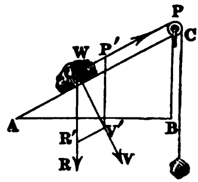
3. Sixteen sailors, exerting each a force of 29 lbs., push a capstan with a length of lever equal to 8 feet, the radius of the capstan being 1 ft. 2 inches. Calculate the resistance which this force is capable of sustaining. *Ans.* 1 ton, 8 cwt. 1 qr. 17 lbs.

4. Supposing them to have wound the rope round the capstan, so as that it doubles back on itself, the radius of the axle is thus increased by the thickness of the rope. If this be 2 inches, how much will the Power of the instrument be diminished?

Ans. By one-eighth; or $12\frac{1}{2}$ per cent.

5. The Resistance of a sluice-gate, to which a cogged rack is attached, amounts to 1 ton; if this be raised by means of a winch and pinion, having for radii 1 ft. 7 inches, and 2.5 inches, respectively, calculate the power which must be applied to the handle, so as just to sustain the Resistance. *Ans.* 294.73 lbs.

4. **The Inclined Plane.**—The inclined plane is a plane which makes an angle with the horizon, which angle is called the inclination of the plane; thus, if AC be an inclined plane, and AB be a horizontal plane, the angle BAC is called the inclination of the plane AC; and if from any point C a perpendicular CB be let fall upon the horizontal plane, the line CB is called the height of the plane, AB is called the base, and AC the length of the plane. It is usual with engineers to measure the inclination of a plane by the ratio which its height bears to its length, i. e. by the sine of the inclination. For example, if CB be the tenth part of AC, the inclination of the plane is expressed by saying that it is one in ten; i. e. a man should walk ten yards up the plane before he would have ascended one yard in vertical height; the inclination of a plane, when so expressed, is sometimes called the gradient. In the inclined plane we have an example of a fixed surface in a machine; this surface will destroy all pressure applied to it in a perpendicular direction, and by means of this consideration we can easily solve the problem of the inclined plane.



This problem may be thus stated: a heavy body being placed upon an inclined plane; given the direction of the force applied to it, to find the magnitude of the Power which will sustain it on the plane.

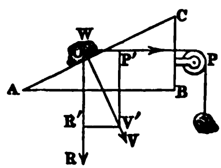
The general case of this problem will be discussed in the examples appended to this section; for general readers it will be sufficient to discuss the two cases which most commonly occur: viz.—the case in which the Power is applied in a direction parallel to the plane, and the case in which the direction of the Power is parallel to the base of the plane.

1st Case.—Let W be a body placed on an inclined plane AC; let a force be applied to sustain it, by means of a cord passing over a pulley P, in the direction OP,

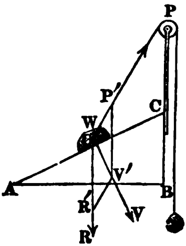
parallel to the plane; and for simplicity let us suppose that the cord is attached to the centre of gravity O, of the body W.

The Power in this case acts in the direction OP, and the Resistance or weight in the vertical direction OR; by Prop. XI. the resultant of these two forces must be perpendicular to the plane AC; let OV be drawn perpendicular to the plane. The problem is therefore reduced to the following: knowing the directions OP and OR of two forces, and the direction OV of their resultant; also knowing one of the forces in magnitude, viz., the weight W; to calculate the magnitude of the other force and resultant. This may be done as follows: let OR′ be measured off, containing as many inches as there are pounds in W, and let the parallelogram R′OP′V′ be completed; then OP′ will represent the Power, and OV′ the resultant, which, by Coroll. Prop. XI. is equal to the pressure on the plane; but the triangle OP′V′ is similar to ACB, because P′OV′ is right, and OP′V′ equal to ACB (Euc. I. 29), therefore OP′ is to P′V′ or OR′, as BC is to AC; or *the Power is to the Resistance as the height of the plane to its length.*

2nd Case.—Let W be a body supported on an inclined plane AC by means of a force acting in the direction OP, parallel to the base of the plane. By what was stated in the preceding case, it appears that the Power and Resistance, and their resultant, act in the known directions OP, OR, and OV; and that the Resistance, or weight W, is known in magnitude; let OR′ be measured off as before, to represent it, and the parallelogram OR′V′P′ completed; then OP′ and OV′ will represent the Power and resultant pressure on the plane AC; but the triangle OP′V′ is similar to ABC: therefore OP′ is to P′V′ or OR′, as CB is to BA; i. e. *The Power is to the Resistance, as the height of the plane to its base.*



EXAMPLES.



Let the force be applied in any direction OP, to sustain the body W on the inclined plane; then, as before, the directions of the Power, Resistance, and Resultant are known, viz., OP, OR, and OV; and the magnitude of the Resistance is known; therefore, the magnitude of the Power and Resultant can be calculated: let OR′ represent the Resistance, then completing the parallelogram OR′V′P′, OP′ and OV′ will represent the Power and Pressure on the plane; but OP′ is to P′V′ or OR′, as sine ROV is to sine POV; and ROV is equal to the inclination CAB, because OR′ and OV are perpendicular to AB and AC respectively; also POV is the angle made by the Power with the perpendicular to the plane; therefore, finally,—*The Power is to the Resistance, as the sine of the inclination is to the sine of the angle made by the Power with the perpendicular to the plane.*

1. A road has a rise of 1 yard in 62; calculate the number of minutes and seconds in its inclination. *Ans.* 55′ 27″.

2. If the inclination of a road be 4°, find a number which shall bear to unity the ratio of the length to the height. *Ans.* 14.33.

3. The longest inclined plane in the world is the road from Lima to Callao, in S. America; it is 6 miles long, and the fall is 511 feet. Calculate the inclination. *Ans.* 55′ 27′, or 1 yard in 62.

4. If the force required to draw a waggon on a horizontal road be $\frac{1}{21}$th part of the weight of the waggon; what will be the force required to draw it up a hill, the slope of which is 1 in 43?

Ans. $\frac{1}{14 \cdot 11}$th part of the weight.

5. If the force required to draw a train of carriages on a level railroad be $\frac{1}{250}$th part of the load; find the force required to ascend a gradient of 1 in 56. *Ans.* $\frac{1}{43 \cdot 75}$th part of load.

6. What force is required (neglecting friction) to roll a cask weighing 964 lbs. into a cart 3 feet high, by means of a plank 14 feet long resting against the cart?

Ans. The force must exceed 206 lbs.

7. Let the angle POV be 101°, and the inclination of the plane

6°; if the weight W be 2 cwt. 1 qr. 17 lbs.—find the power necessary to sustain it. *Ans.* 28.64 lbs.

8. If the angle POV be 120°, the inclination 2°, and the weight 31 tons; find the power. *Ans.* 1 ton, 4 cwt. 3 qrs. 26 lbs.

9. Let the angle POV be 167°, and the inclination 1 in 26, the weight being 22 tons; find Power. *Ans.* 3.761 tons.

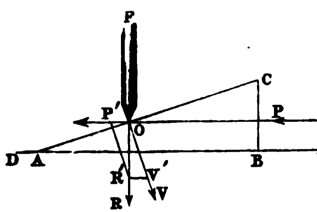
10. If a waggon weighing 2 tons, 14 cwt., rest upon an incline of 1 in 27; find the pressure upon the road.
Ans. 2 tons, 13 cwt., 3 qrs., 23.8 lbs.

11. If the force required to overcome friction at a given speed upon a horizontal railroad be 10 lbs. per ton, and the similar force upon a turnpike road 236 lbs. per ton; find the effect produced upon both by a gradient of 1 in 21, and show why the gradients on railroads should be much less than on common roads.

Ans. The force on railroad is increased to 116.6 lbs. per ton.
——— turnpike ,, 342.6 lbs. per ton.

Hence, it appears that the force on the railroad should be increased more than eleven times, while the force on the turnpike road should be increased less than one and a half times.

5. **The Moveable Inclined Plane or Wedge.**—The inclined plane, considered in section 4, was a fixed plane of indefinite dimensions: we shall now consider an inclined plane of finite dimensions, placed upon a horizontal plane, upon which it is capable of sliding. Let DE be the horizontal plane on which the moveable inclined plane ABC is placed. Let F be a rod through which the Resistance is applied to the plane at the point O, and let us suppose, for simplicity, that the Power is applied to the back of the plane in the direction PO, parallel to the base AB. In this system we have two surfaces, DE and AC, one fixed and the other moveable, but each capable of sustaining pressure in a perpendicular direction only. The Re-

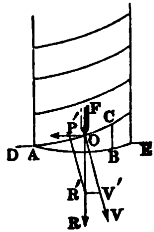


sistance acting through the rod F must therefore produce its effect upon the plane in the direction OV; while the pressure resulting from the Power and Resistance must be sustained by the fixed plane DE, and act in the direction OR. We know, therefore, the directions of the Power, and of the two pressures on the planes AC and DE; and we also know the magnitude of the Power; we can therefore calculate the magnitude of the pressures on AC and DE as follows: Measure off the line OP′, containing as many inches as there are pounds pressure in the Power, and complete the parallelogram P′OV′R′; OV′ and OR′ will represent in magnitude the pressures on the planes AC and DE. But the triangle P′OR′ is similar to ABC, and therefore OP′ is to OR′ as BC is to BA; i. e. *The Power is to the Pressure on the horizontal plane as the height of the moveable plane is to its base.* In like manner it appears that OP′ is to OV′ as BC is to CA; i. e. *The Power is to the Pressure on the moveable plane as the height of the plane is to its length.*

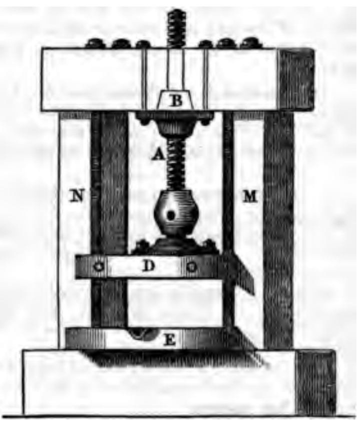
The moveable inclined plane, which has just been described, is the wedge, provided the Power applied be a pressure, and not a blow; but as the latter is the practical method of using a wedge, and as the theory of the moveable inclined plane does not apply to such forces as impacts, we shall omit the consideration of the wedge, when acted on by impact, and proceed to a most important example of the moveable inclined plane, which is called—

6. **The Screw.**—The screw is a moveable inclined plane wrapped round a circular cylinder. In order to explain the mode of its action, we shall suppose the figure in p. 51 to be wrapped round a vertical cylinder. The base DE being fixed, if a Power OP′ tending to turn the screw round be applied, the Resistance will react at the point O in two directions, with pressures measured

by OV′ and OR′; the latter being the pressure sustained by the horizontal plane; and the former, the pressure perpendicular to the thread of the screw; hence, as before, OP′ is to OR′ as CB is to BA; but when the plane is wrapped completely round the cylinder, the interval between two threads will be to the circumference of the cylinder as CB is to BA. Hence, *The Power applied at the circumference of cylinder is to the vertical Resistance, as the interval between the threads is to the circumference of the cylinder.* Take, for example, the screw-press; resistances similar to F are applied at each point of the thread of the screw A, by means of a nut B, which is tapped



with a screw that fits into the thread cut on the cylinder; this nut, together with the uprights M and N, supports the resistance caused by the substance pressed between the plates D and E, and the power is applied at the extremity of a lever attached to the screw. This power bears to the power at the circumference of the screw the same ratio that the radius of the screw bears to the radius of the circle described by the Power, on the principle of the wheel and axle; or the same ratio that the circumference of the screw bears to the circumference of the circle described by the Power. Hence, the Power is to the Resistance parallel to the axis of the screw, in



a ratio compounded of the ratio of the interval between the threads to the circumference of the cylinder, and of the circumference of cylinder to the circumference of circle described by Power; i. e. *The Power is to the Resistance parallel to the axis, as the interval between the threads is to the circumference of circle described by the Power.*

EXAMPLES.

1. *To find the ratio which the Power bears to the Pressure perpendicular to the thread of a Screw.*—It appears from section 5, that the Power is to the Pressure, on the moveable inclined plane, as the height to the length of plane; hence, in the screw (fig. p. 53), P′O is to OV′ as CB is to CA; or as the interval between the threads is to the length of thread in one turn of the screw. Hence, the Power applied at the circumference of cylinder is to the pressure on the thread, as the interval between the threads is to the length of thread; this ratio must be compounded with the ratio of the Power at O, to the Power at the circumference of winch. Hence,—

The Power is to the Pressure on the thread, in a ratio compounded of the ratio of the interval between the threads to the length of thread in one turn of the screw; and of the ratio of the radius of the screw to the radius of the circle described by the Power.

2. If the interval between the threads of a screw be $\frac{1}{11}$th of an inch, and the circumference of the circle described by the Power be 2 feet; what Resistance will a Power of 13 lbs. sustain?

Ans. 1 ton, 10 cwt. 2 qrs. 16 lbs.

3. If the interval be $\frac{1}{8}$th of an inch, and the diameter of the circle described by the Power be 2 feet; find what Resistance a Power of 17 lbs. will sustain. *Ans.* 5 tons, 3 cwt.

4. If the interval be $\frac{3}{8}$ths of an inch, and the radius of the circle described by the Power be 2 ft.; find Resistance which a Power of 32 cwt. will sustain. *Ans.* 643 tons, 8 cwt.

5. If there be 19 turns of a screw in $1\frac{5}{8}$th inch, and if the handle, 1 ft. 5 in. in length, be worked with a Power of 2 cwt. 1 qr. 17 lbs.; find the Resistance. *Ans.* 149.9 tons.

6. The interval between the threads is $\frac{1}{6}$th of an inch, and the diameter of the cylinder is 1 inch, and a Power equal to 139 lbs. acts in a circle whose circumference is 3 feet; find the Pressure on the

thread of the screw. The Pressure is, by Example 1, equal to the Power, multiplied by the ratio of length of the thread to the interval between the threads, and of the circumference of the circle described by Power to the circumference of the cylinder; i. e. equal to 139 lbs. multiplied by $\frac{3·1460}{0·1666}$ (the numerator of this fraction is the length of the thread, which is equal to the square root of the sum of the squares of the interval between the threads and of the circumference of the cylinder), multiplied by $\frac{36}{3·14159}$, equal to *Ans.* 30066 lbs.

7. The interval between the threads of a screw being $\frac{1}{11}$th of an inch, and the diam. of cylinder $\frac{3}{8}$ths of inch; find length of thread in 14 revolutions. *Ans.* 16.512 inches.

8. The interval being $\frac{1}{8}$th inch, and the diam. $\frac{15}{8}$ inch; find length of thread in one revolution. *Ans.* 5.90 inches.

9. A pressure of 5 tons is exerted at the back of a moveable inclined plane, having an inclination of 2°; find the weight which this force will sustain. *Ans.* 143 tons, 3 cwt. 2 qrs. 21 lbs.

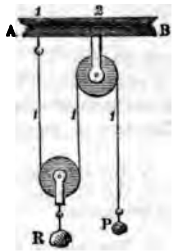
10. Find, in this case, the pressure sustained by the inclined plane. *Ans.* 143 tons, 5 cwt. 1 qr. 8 lbs.

7. **The Pulley.**—A force may be transmitted from one point to another by means of a cord, and if there were no friction, this force might be transmitted undiminished in any direction. In order to diminish friction in thus transmitting a force from one point to another, by means of a cord, we make the cord pass over a wheel with a groove cut in its circumference, which turns with the cord, and so diminishes the friction. This wheel is called the *sheaf*, and the wooden frame in which it works, and to which its axle or pivot is fastened, is called the *block* of the pulley. The principle on which the equilibrium of a system of pulleys is determined is the following:—

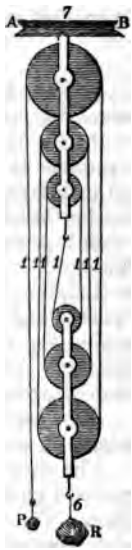
The tension of each cord employed in the system is constant, and equal to the Power applied at its extremity.

The *single moveable Pulley* is represented in the annexed figure, which is intelligible without any description. The power applied at P produces a constant tension through the whole cord; this is denoted by the

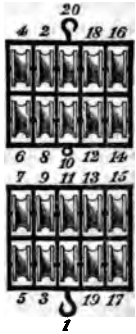
figure I placed beside each portion of the cord. The weight R is attached to the moveable pulley, and since the block of this pulley is supported by two tensions, parallel in direction, and each equal to the Power, it is plain that this block is capable of sustaining a weight equal to twice the Power applied. The fixed beam AB must support, by Prop. XI., a strain equal to the resultant of the Power and Resistance; and since these are parallel in direction, the strain on the beam will be three times the Power. This is shown by the figures annexed to the points at which the cord and fixed pulley are attached, which express the strain on each of these points.



The single moveable pulley may be extended by placing in each block several sheaves, and passing the cord over the upper sheaf in the fixed block, then under the lower sheaf in the moveable block, and so on, as represented in the figure. The tension of the string being constant and equal to the Power, and the moveable block supported by six tensions, we shall have the Resistance sustained equal to six times the Power; and in general, where there are two blocks, one fixed, and the other moveable, *the Resistance or weight supported will be equal to the Power multiplied by twice the number of sheaves in the block.*

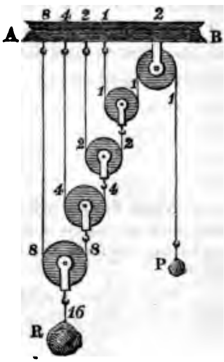


The pulley invented by Smeaton is of this kind; its two blocks are represented in the annexed figure. The cord is attached to the upper block at the point marked 10, it passes under the sheaf 1, then over the



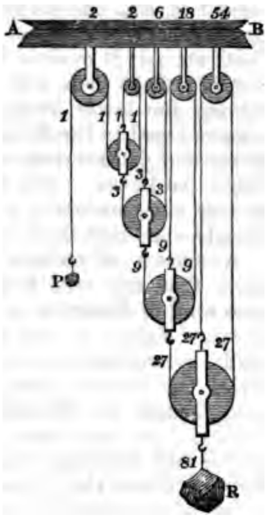
sheaf 2, 3, &c., and the Power is applied at its extremity after it has passed over the sheaf 20. It is evident that in this system of pulleys there will be 20 cords sustaining the lower block, having each a tension equal to the Power applied. The Resistance is therefore equal to twenty times the Power. The fixed beam will, in this case, sustain a pressure equal to twenty-one times the Power.

A system of pulleys which contains more than one cord is called a *Burton*. One kind of Burton is represented in the annexed figure, in which there are four cords, sustaining tensions, represented by 1, 2, 4, and 8 times the Power respectively; the weight sustained being equal to 16 times the Power. As the sum of the Power and Resistance is therefore 17 times the Power, this must be sustained by the beam AB; the manner in which this is done is represented by the numbers placed over the beam, which express the strain produced at each point of attachment.



Combinations of pulleys, such as we have here described under the name of Burtons, are frequently called by modern English mathematical writers, systems of pulleys; of these systems of pulleys, or Burtons, it is customary to describe three, of which two are mentioned in the text, and the third is described in Exercise 4, appended to the present chapter. There are other Burtons, of a simpler character, which we have not room to specify, such as the Spanish Burton, used for a long time in the moving of heavy casks of wine, and other goods.

Another kind of Burton is represented in the annexed figure, in which also there are four cords; but each cord, instead of being fastened to the beam AB, is made to pass over a fixed pulley, and is then attached to the upper part of its own block. The tensions supported by these cords are 1, 3, 9, and 27 times the Power respectively. The weight supported is 81 times the Power. The numbers above the beam AB denote the strain on each point of support; their sum is equal to 82, i. e. equal to the sum of the Power and Resistance.



EXAMPLES.

1. What Power will be required in a Burton of the first kind with four moveable pulleys to sustain a weight of 17 tons, 12 cwt., neglecting friction, the rigidity of the cord, and the weight of the blocks. *Ans.* 1 ton, 2 cwt.

2. What Power would be required in a Burton of the second kind with 4 moveable pulleys to sustain the same weight? *Ans.* 4 cwt. 1 qr. 10 lbs.

3. If a Burton of the first kind contain 11 pulleys, what weight will 13 lbs. support by means of it? *Ans.* 11 tons, 17 cwt. 2 qrs. 24 lbs.

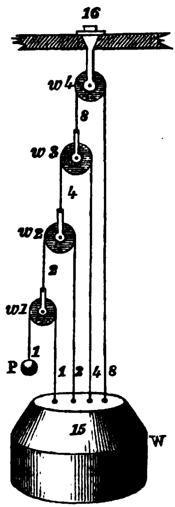
4. A Burton of the second kind contains 7 pulleys; what weight will 17 lbs. support by means of it? *Ans.* 16 tons, 11 cwt. 3 qrs. 23 lbs.

5. Compare the efficiency of Smeaton's block with a Burton of the second kind with 5 moveable pulleys. *Ans.* 12.15 to 1.

6. Compare the efficiency of the two kinds of Burtons, each having 10 moveable pulleys. *Ans.* 57.6 to 1.

The machines which have been described in this chapter are among the simplest and most useful of those which occur in practice. There are many combinations of these machines with each other and with other machines, which could not be described without entering into details unsuited to the limits we have prescribed to ourselves in this work. Those which have been described are sufficient to illustrate Prop. XI., and to show the method to be followed in investigating the relation between the Power and Resistance in other cases.

Exercises on the Equilibrium of Machines.



1. Find the ratio of the Power to the Weight in the Burton of the first kind, with n cords.

$$Ans. \quad \frac{P}{W} = \frac{1}{2^n}.$$

2. Find the ratio of the Power to the Weight in the Burton of the second kind, with n cords.

$$Ans. \quad \frac{P}{W} = \frac{1}{3^n}.$$

3. If, in the first kind of Burton, there be n moveable pulleys, whose weights are w_1, w_2, w_3, ... w_n; find the relation of the Power to the Weight.

$$Ans. \quad P \times 2^n = W + 2^{n-1}w_1 + 2^{n-2}w_2 + \&c. \ldots . \; 2w_{n-1} + w_n.$$

4. If the first kind of Burton be supposed inverted, as in the annexed woodcut, we shall have another system of pulleys consisting of different cords, each of which terminates in the Weight; what is the ratio of the Power to the Weight in such a system, n being the number of distinct cords?

$$Ans. \quad \frac{P}{W} = \frac{1}{2^n - 1}.$$

5. If the weights of the pulleys be taken into account, what is the relation of the Power to the Weight?

$$Ans.\ W = P(2^n - 1) + (2^{n-1} - 1)w_1 + (2^{n-2} - 1)w_2 + \&c. \ldots + w_{n-1}.$$

6. If the weights of the pulleys be taken into account in the second kind of Burton; find the relation of the Power to the Weight.

$$Ans.\ 3^n \times P = W + 3^{n-1}w_1 + 3^{n-2}w_2 + \&c.\ 3w_{n-1} + w_n.$$

7. If a power be in equilibrium with a weight, by means of several levers, the one acting upon the other; if a, a', a'', &c. denote the arms of the levers lying next the power, and b, b', b'', &c. the arms lying next the weight; find the ratio of the Power and Weight.

$$Ans.\ \frac{P}{W} = \frac{b \times b' \times b'' \ldots}{a \times a' \times a'' \ldots}.$$

8. If a power be in equilibrium with a weight, by means of a number of toothed wheels; prove that the Power is to the Weight as the continued products of the diameters of the pinions is to the continued product of the diameters of the wheels.

9. In a single moveable pulley, in which the two directions of the string are not parallel, prove that the Power is to the Weight as the radius of the pulley is to the chord of the arc in contact with the string.

10. In a Burton of the first kind, in which the strings are not parallel, the Power is to the weight as the continued product of the radii of the pulleys to the continued product of the chords of the arcs in contact with the strings.

11. A weight of 21 cwt. 1 qr. 20 lbs. is to be lifted by means of a crane, of which the axle for receiving the rope is a foot in diameter, and carries a wheel of 50 teeth, worked by a pinion of 6 teeth. What force must a man apply to the handle of the winch by which the pinion is worked, and of which the arm is 16 in. long, in order to raise the weight. *Ans.* 108 lbs.

12. If two bodies resting on two opposite inclined planes, and connected by a cord passing over a pulley at their common summit, are in equilibrium, prove that the pressures on the planes are inversely as the tangents of their inclinations.

13. What horizontal force is necessary to support a weight of 50 lbs., on a plane inclined at an angle of 15° to the horizon?

Ans. 13.4 lbs.

14. In a balance with unequal arms, the true weight of any commodity is a geometric mean between its apparent weights when placed in the two scales.

15. Supposing that $5\frac{1}{2}$ turns cause the head of a screw to advance two-thirds of an inch, what power applied at the extremity of an arm 18 inches long will be required to produce a pressure of 1000 lbs. upon the head of the screw? *Ans.* 1.07 lbs.

16. The arms of a false balance are to each other as 7 to 8, and the weight is put into one scale as often as into the other. What will be the gain or loss per cwt. to the seller?

Ans. Loss of 1 lb. per cwt.

17. In a system of 4 distinct cords, in which each cord is attached to the weight, determine the weight supported, and the strain upon the fixed pulley, the power being 100 lbs., and the weight of each pulley 5 lbs. *Ans.* Weight $= 15P + 11w = 1555$ lbs.
Strain $= 16P + 15w = 1675$ „

18. A system of three similar levers in combination is employed to test the strength of a chain cable. Find the ratio of the arms, upon the supposition that every ounce in the scale-pan of the last indicates a strain of a cwt. all but 4 oz. upon the chain.

Ans. $r = 12.137$ lbs.

19. In a system of pulleys, where each cord is attached to the weight, there are three moveable pulleys, each weighing $2\frac{1}{2}$ lbs. What power is required to support a weight of 6 cwt.?

Ans. 95.43 lbs.

20. Find the power that will support a weight of 100 lbs. by means of a system of 4 pulleys, the strings being all parallel, and all attached to the weight, each pulley weighing 1 lb. *Ans.* $5\frac{14}{15}$ lbs.

21. In a similar system of three pulleys, all the strings are attached to a bar at equal distances of 2 inches (which is equal to the common radius of the pulleys). Find from what point of the bar the weight must be suspended, so that the bar may continue horizontal, neglecting the weights of the pulleys.

Ans. At a distance of $2\frac{6}{7}$ in. from the attachment of the first cord.

22. Where must the fulcrum be placed in order that a man who can lift a weight of 120 lbs. may, with a heavy crowbar (which weighs 30 lbs., and is 5 ft. long), just raise 5 cwt.?

Ans. The greater arm is $4\frac{7}{142}$ ft.

23. A handle, with an arm 2 ft. long, turns an endless screw, which works a wheel with 60 teeth; and a chain, supporting a weight of 5 tons, is coiled up on the shaft of this wheel, the shaft being 6 inches in diameter. What force must be applied to the handle to balance this weight *Ans.* $\frac{1}{96}$th of a ton, or $23\frac{1}{3}$ lbs.

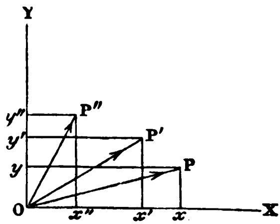
CHAPTER IV.

ON THE EQUILIBRIUM OF FORCES IN A PLANE.

1. Equilibrium of two forces in a plane.—2. Equilibrium of three or more forces meeting at a point.—3. On the transference of forces in a plane.—4. On the equilibrium of pairs or twists.—5. Equilibrium of three or more forces in a plane, not meeting in the same point.

1. **Equilibrium of two Forces in a Plane.**—If two forces lie in the same plane, they must either meet in a point, or be parallel to each other. 1st. If they meet in a point, they will always have a resultant, by Prop. I.; and this resultant will not be zero unless the forces be *equal to each other, act in the same line, and in opposite directions.* 2nd. If the forces be parallel to each other, it appears from Prop. V., that they will always have a resultant, and that this resultant will never be zero, except in the case of two *equal, opposite, and parallel* forces, and that in this case, its point of application is infinitely distant, and the *pair* of forces has no resultant, properly so called, but becomes a Twist, or Couple, or Moment, as it is called by various writers on Mechanics.

2. **Equilibrium of three or more Forces meeting at a Point.**—Let any number of forces P, P', P'', &c., represented by the lines OP, OP′, OP″, &c., meet in a point O; required to find the conditions of their equilibrium. Through O draw two lines OX and OY at right angles to each other,



and from the points P, P′, P″, &c., let fall perpendiculars Px, Py; P′x', P′y'; &c. The components of the forces P, P', &c., in the direction OX, are Ox, Ox', &c.; and in the direction OY, their components are, Oy, Oy', &c.; and since these components are at right angles to each other, and therefore exert no mutual influence, it is necessary for equilibrium that their respective algebraic sums should be equal zero; therefore,

$$o = Ox + Ox' + Ox'' + \&c.$$
$$o = Oy + Oy' + Oy'' + \&c.$$

If we denote by α, β, the angles POX, POY; by α', β', the angles P′OX, P′OY, &c.; we have for the two necessary conditions of equilibrium,

$$o = P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \&c.$$
$$o = P \cos \beta + P' \cos \beta' + P'' \cos \beta'' + \&c.$$

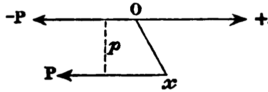
Hence we obtain the following rule for finding the equilibrium of any number of forces in the same plane meeting at a point.

Resolve the forces in any direction, and equate the sum of their components to zero; and then resolve the forces in a direction at right angles to the former direction, and equate to zero the sum of the components thus found; the two equations so determined contain all the conditions of equilibrium.

3. **On the Transference of Forces in a Plane.**—If a force be given acting at a certain point, it is often convenient to consider its effect at some other point not situated in its direction. In order to consider its effect at this other point, it is necessary to examine how a force may be transferred from one point to another of a plane; this is done as follows:—

Let P be a force acting at any point x, and let it be required to ascertain its effect at any other point O. We

may introduce at O two forces $+P$ and $-P$, parallel to the original force, and equal and opposite to each other; for, since these forces equilibrate each other, they do not influence the effect of the original force P.



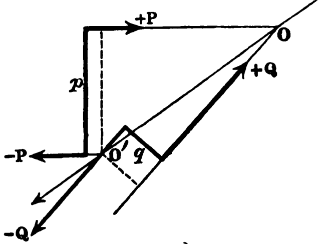
It is evident from the figure that the effect of the original force P on the point O is the same as a force P applied at that point, together with a pair P of equal, opposite, and parallel forces acting at a distance p from each other. Hence,

The effect of any force at a point not situated in its direction, is equivalent to an equal force applied at that point, together with a Twist formed of two equal forces, whose moment is Pp, at a distance from each other equal to the perpendicular let fall from the point on the original force.

4. **On the Equilibrium of Twists.**—The equilibrium of Twists is established by the following Proposition:—

PROPOSITION XII.—THEOREM.

If a Pair of forces, or a Twist $\pm$ P *be situated at a distance* p, *and another Pair or Twist* $\pm$ Q *be situated at a distance* q, *and the Twists be, one right-handed and the other left-handed; they will equilibrate each other, provided* P $\times$ p *equal* Q $\times$ q.



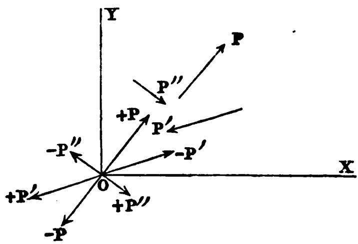
For, let the forces $+P$ and $+Q$ meet at O, and the forces $-P$ and $-Q$ meet at O′; if $+P$ and $+Q$ be compounded at O, their resultant will pass through O′; for letting fall perpendiculars from O′ on $+P$ and $+Q$, these perpendiculars are equal to p and q, and since $Pp = Qq$, by Prop. III., the point O′ is situated on the Resultant of $+P$ and $+Q$. Compounding in like manner the forces $-P$ and

$-Q$ at O′, it can be proved that their resultant passes through O; therefore the line OO′ is the common direction of the resultants of $+P$, $+Q$, and of $-P$, $-Q$; but these resultants are equal in magnitude, because $\pm Q$ are parallel, and also $\pm P$ are parallel; and therefore the two resultants destroy each other, since they act in the same line, are equal in magnitude, and opposite in direction. Therefore the Twists $P \times p$ and $Q \times q$ equilibrate each other.—Q. E. D.

COROLLARY.—Hence the effect of any number of Twists in the same plane, Pp, $P'p'$, &c., is found by taking the algebraic sum of the products Pp, $P'p'$, &c.; or

$$T = Pp + P'p' + P''p'' + \&c.$$

5. **Equilibrium of three or more Forces in a Plane, not meeting in the same Point.**—Let any number of forces P, P', P'', &c., act in the same plane, and not meet in the same point; the conditions necessary for their equilibrium may be thus found.



Take any point O in the plane, and at O introduce forces $\pm P$, $\pm P'$, $\pm P''$, &c., as in section 3; we shall thus have the original forces equivalent to forces P, P', P'', &c., acting at the point O, together with the Pairs or Twists Pp, $P'p'$, &c., generated by the transference of the forces to the point O.

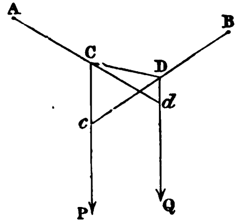
The equilibrium of the entire system is therefore reduced to the equilibrium of the forces P, P', &c., acting at the point O, and the equilibrium of the Twists Pp, $P'p'$, &c.

Hence, by equation (3), section 2, and Corol. of Prop. XII., we find the three following conditions of equilibrium, using any two axes OX and OY, at right angles to each other.

$$0 = P \cos a + P' \cos a' + \&c.$$
$$0 = P \cos \beta + P' \cos \beta' + \&c.$$
$$0 = Pp + P'p' + P''p'' + \&c. \qquad (4)$$

MISCELLANEOUS PROBLEMS.

1. A string ACDB, of which the extremities A, B are fixed, supports the weights P and Q, suspended at knots C and D; if AC, BD be produced to meet the directions of the weights in d, c respectively; show that

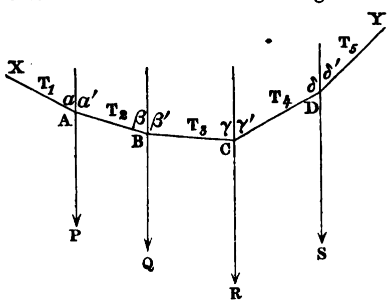


$$P : W :: Dd : Cc.$$

2. Also, prove that the strains on the fixed points A and B, are proportional, on the same scale, to the lines Cd and Dc.

3. A string XABCDY, of which the extremities X and Y are fixed, has weights P, Q, R, S, attached at the points A, B, C, D.

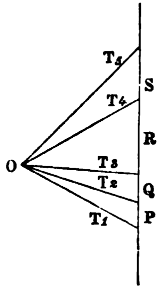
This system forms what is called a Funicular Polygon; prove that if T_1, T_2, T_3, T_4, T_5, denote the tensions of the several strings which form the sides of the polygon, they are proportional to the secants of the angles made with the horizon by each side; and that the weights P, Q, R, S, are proportional to



$\cot a + \cot a'$:
$\cot \beta + \cot \beta'$:
$\cot \gamma + \cot \gamma'$:
$\cot \delta + \cot \delta'$.

Ans. For, draw any vertical line, and assume any point O, and draw OT_1, OT_2, &c., parallel to the sides of the polygon, to meet the assumed vertical line.

Since the point A is held in equilibrium by the tensions T_1, T_2, and the weight P, and the point B is held in equilibrium by the tensions T_2, T_3, and the weight Q, and so on; it follows that the sides of the triangles having their vertices at O, and their bases on the vertical line, represent the equilibrating forces; the sides T_1, T_2, &c., representing the tensions, and the bases P, Q, R, &c., representing the weights; but it is evident that the sides are the secants of the angles made with the horizontal line, and the bases are proportional to the sums of the cotangents of the base angles, which are α, α', β, β', &c., in each triangle.



4. A sphere, whose weight is W, rests on two inclined planes, whose inclinations are i and i'; find the pressure on each plane.

Ans. The pressure on plane $(i) = \dfrac{W \sin i'}{\sin (i + i')}$.

The pressure on plane $(i') = \dfrac{W \sin i}{\sin (i + i')}$.

5. One end of a ladder, whose weight is W, is prevented from sliding by a peg fastened in a horizontal plane, and the other extremity rests against a smooth vertical wall; find the pressure on the wall, and the direction and magnitude of the pressure on the fixed peg. Let m and n be the segments into which the centre of gravity divides the ladder, θ the angle made with the wall, P the pressure on the wall, and Q the pressure on the peg.

Ans. $P = W \dfrac{m}{m+n} \tan \theta.$

$$Q = W \sqrt{1 + \frac{m^2 \tan^2 \theta}{(m+n)^2}}.$$

The direction of Q is found by drawing a vertical line through the centre of gravity to meet the perpendicular from the wall at the top of the ladder, and from the intersection of these lines, drawing a line through the foot of the ladder.

6. A trap door turning on a hinge is supported by a weight P attached to a string passing over a pulley placed at the other side of the door in the same horizontal plane; find its position of equilibrium.

Let W be the weight and l the length of the door, and let θ be the angle made by it with the horizon.

$$P = W \frac{\cos\theta}{2\cos\frac{1}{2}\theta}.$$

7. If a beam rest on two inclined planes whose inclinations are i, i' respectively; find the pressure on each plane, and the angle made by the beam with the horizon when in equilibrium.

Since the reactions of the planes at the extremities of the beam are perpendicular to the planes, and must meet on the vertical drawn through the centre of gravity of the beam; we find

$$\textit{Ans.}\quad P = \frac{W \sin i'}{\sin(i+i')}.$$

$$P' = \frac{W \sin i}{\sin(i+i')}.$$

$$\tan\theta = \frac{\sin(i'-i)}{2\sin i \sin i'}.$$

8. A roof ACB is wholly composed of beams forming isosceles triangles of which AB is the base; find the horizontal thrust on the side walls.

If W be the weight of the roof, and ϕ the angle which it makes with the horizon, it is easy to see, by means of Example (5), that

$$\text{The Horizontal Thrust} = \tfrac{1}{4} W \cot\phi.$$

DYNAMICS.

CHAPTER I.

DEFINITIONS AND LAWS OF MOTION.

1. Motion, or Velocity.—2. Quantity of Matter and Motion.—3. Composition of Velocities.—4. Laws of Motion.

Dynamics is that branch of Mechanics which treats of Motion, considered as an effect of force. In Statics we compared forces by means of the pressures produced by them; in Dynamics we are to compare forces by means of the motions produced; the first thing to be done is, therefore, to explain what is meant by motion or velocity.

1. **Motion, or Velocity.**—The motion or velocity of a body is the rate at which the body is moving, and is measured by the space passed over in a given time; such as miles per hour, feet per second, &c.

In order to have a uniform unit of velocity, it is customary to express velocities in feet and seconds; and when velocities are expressed in any other way, they should be reduced to their equivalent value in feet and seconds. The unit velocity is, therefore, the velocity of a body which is moving at the rate of one foot in a second. Velocities are either uniform or variable. A uniform velocity is one in which the velocity is constant at each instant; a variable velocity is one in which the velocity, measured at different instants, is different. The velocity of a body, at any instant, is measured by finding the number of feet which would be described in

a second by a body supposed to move uniformly at the rate at which the given body is moving at the given instant.

Let V represent the velocity of a body moving uniformly; i. e. the number of feet it describes in one second; and let S denote the number of feet passed over in T seconds. Then,

$$V : S :: 1 : T;$$

i. e. the space described in one second : the space described in T seconds :: 1 second : T seconds. Hence (since the rectangle under the means is equal to the rectangle under the extremes),

$$S = VT. \tag{1}$$

This equation contains three quantities,—space, time, and velocity: any two of these being given, we can calculate the third.

EXAMPLES.

1. A body moves at the rate of 754 yards per hour. Find the velocity in feet per second. *Ans.* 0.628.

2. A railway train travels at the rate of 40 miles per hour. Find its velocity in feet per second. *Ans.* 58.66.

3. A train takes 7 h. 31 m. to travel 200 miles. Find its velocity. *Ans.* 39.02 ft. per sec.

4. Supposing the diameter of the earth at the equator to be 7925 miles, and that it revolves once in 23 h. 56 m. Find the velocity of a point at the equator. *Ans.* 1525.7 ft. per sec.

5. Sound travels at the rate of 1090 feet per second. If a gun be fired at Holyhead, how long would the sound take to reach Dublin, the distance being 64 miles? *Ans.* 5 m. 10 s.

6. A railway train sets out to perform a journey of 347 miles, and travels at the rate of $30\frac{1}{4}$ miles per hour; after travelling 261 miles, an accident happens to the machinery, which stops the train 2 h. 27m., and renders the locomotive incapable of travelling at a rate of more *than 16 miles* per hour. Find the time occupied by the whole journey. *Ans.* 16 h. 26 m.

7. A steamer going at the rate of 12 knots per hour steams for 4 days, and then alters her velocity to 13 knots per hour, at which she continues for 3 days 7 hours. Find the distance in miles she has travelled at the end of the whole time. *Ans.* 2507 miles.

Note.—A knot, or geographical mile, is the sixtieth part of a degree of the meridian, and contains 6076 ft. in lat. 45°.

Having defined velocity or motion, we can now explain the mode of estimating a force *dynamically;* in Statics we estimated the magnitude of a force by the number of pounds or ounces it was capable of supporting; in Dynamics we estimate the magnitude of a force by the *velocity it is capable of producing in a given time,* such as a second. We suppose the force to act constantly upon the body originally at rest, for the space of one second, and then to cease; the rate or velocity at which the body is then moving is the dynamical measure of the force.

2. **Quantity of Matter and of Motion.**—The following definitions should be carefully committed to memory:—

DEFINITION.—The *specific gravity,* or density of a body, is *the number which expresses the ratio which the weight of a cubic inch of the body bears to the weight of a cubic inch of distilled water, at the temperature of* 60° *Fahrenheit.*

DEFINITION.—*The quantity of matter* contained in a body is the *product of its specific gravity and volume* expressed in cubic inches, feet, &c. The quantity of matter is also called the *mass of a body.*

DEFINITION.—*The quantity of motion* in a body is the product of its *quantity of matter, and the velocity with which all its parts are moving.* The quantity of motion is also called the *momentum of a body.*

The method of finding the specific gravities of bodies,

and a Table of specific gravities, are given in the Manual of Hydrostatics. Here it is sufficient to give the definition of specific gravity, and to add a few examples to illustrate the definitions of quantity of matter and motion.

EXAMPLES.

1. The specific gravity of lead and copper being 11.35 and 8.90, respectively (i. e. the weight of a cubic inch of water at 60° F. being unity, the cubic inch of lead and copper will weigh respectively 11.35 and 8.90); find the ratio of the quantities of matter contained in a cubic foot of solid lead and a cubic yard of solid copper.
Ans. 0.04723.

2. The specific gravities of hammered gold and silver being 19.35 and 10.51, find the ratio of the quantities of matter contained in a bar of gold measuring 4.17 inches long, by 0.64 inches wide, by 0.31 inches deep; and a bar of silver measuring 13.22 inches long, by 1.14 inches wide, by 0.65 inches deep. *Ans.* 0.15549.

3. A bar of solid silver (sp. gr. 10.51) moves with a velocity of 5 miles per hour, its dimensions being the same as in the last example. Find its quantity of motion, the volume being measured in cubic inches. *Ans.* 755.012.

4. A cubic foot of copper (sp. gr. 8.9c) moves with a velocity of 1407 yards per minute. Find its quantity of motion, volume being measured in cubic inches. *Ans.* 1081926.72.

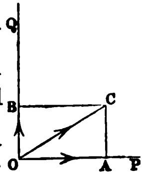
5. A block of Carrara marble (sp. gr. 2.716) measuring 2 yards long, by 2 feet wide, by 1.16 feet deep, in falling from a height acquires a velocity of 13 feet per second. What is its *momentum*, volume being measured in cubic feet? *Ans.* 491.48.

6. What should be the velocity of a cubic foot of lead (sp. gr. 11.35), in order that its momentum should be equal to that of the marble block in the last example?
Ans. 43.302 feet per second.

7. A bullet of lead, containing 0.267 cubic inches, is projected from a gun. Find with what velocity it should move, in order that its momentum should be equal to the momentum of a ball of copper, containing 13.47 cubic inches, moving at the rate of 14 feet per minute. *Ans.* 9.230 feet per second.

3. **Composition of Velocities.**—It has been ascertained by numerous observations and experiments, that

velocities may be compounded by the same laws as statical forces or pressures; and the composition of velocities has been even considered by some metaphysical writers on Mechanics as a consequence of our elementary conceptions of Time and Space. Without pledging ourselves to such subtleties, it will be sufficient here to explain what we mean by the Composition of Velocities, considered simply as a matter of fact and observation.



If a body placed at O move in the direction OP, with a velocity which would bring it to the point A in one second; and at the same time move in the direction OQ with a velocity which would bring it to the point B in one second; the effect of both velocities acting jointly will be to bring the body, at the end of one second, to the point C, found by completing the parallelogram OACB. The truth of this fundamental law of motion is exhibited in many familiar cases; for example, on board of a ship moving equably the movements of the sailors are not influenced by the motion of the ship: it is certain, also, from the facts observed by astronomers, that the earth is in motion; nevertheless, all motions take place on its surface independently of that motion, and as if it were at rest.

It appears, therefore, that *Velocities* are compounded by the same law as *Pressures* and that all the results of Statics may be transferred to Dynamics, by substituting velocities for pressures.

This identity in the laws of Composition of Velocities and Pressures is only natural; for since velocity and pressure are only different effects of the same Force, it is natural to expect that, to a great extent, they will be governed by the same laws.

EXAMPLES.

1. A body moves under the influence of two velocities, at right angles to each other, equal respectively to 17.14 feet and 13.11 feet per second. Find the magnitude of the resultant motion.
Ans. 21.579 feet per second.

2. Find in the same case the angles into which the resultant divides the right angle. *Ans.* $37^\circ\ 25'$ and $52^\circ\ 35'$.

3. A ship sails due north, at the rate of 4 knots per hour, and a ball is rolled towards the east, across her deck, at right angles to her motion, at the rate of 10 feet per second. Find the magnitude of the real motion of the ball, compounded of these two motions.
Ans. 12.07 feet per second.

4. Find the direction of the real motion. *Ans.* 56° East of N.

5. A body moves, under the influence of two velocities, at the rate of 36.14 feet per second, and one of the component velocities is 14.31 feet per second. Find the other, on the supposition that they act at right angles. *Ans.* 33.18 feet per sec.

6. If the component velocities be 14.37 feet and 19.22 feet, and make an angle of 22°, find resultant velocity. *Ans.* 33 feet per sec.

7. Find the angles into which the resultant in the last example divides the angle between components. *Ans.* $12^\circ\ 36'$ and $9^\circ\ 24'$.

8. The component velocities are 13.61 feet per second, and 14.62 yards per minute, and make an angle of 41°. Find magnitude of resultant velocity. *Ans.* 14.17 feet per sec.

9. Find the angles into which it divides the angle between components. *Ans.* $39^\circ\ 4'$ and $1^\circ\ 56'$.

10. A boat is rowed across a river (which is $1\frac{1}{4}$ miles wide) in a direction making an angle of 87° with the bank. The boat travels at the rate of 5 miles per hour, and the river runs at the rate of 2.3 miles per hour. Find at what point of the opposite bank the boat will land, if the angle of 87° be made against the stream.
Ans. 898 yards down the stream from the opposite point.

11. In the same case, if the angle of 87° be made with the stream, find the point of landing.
Ans. 1129 yards down the stream from the opposite point.

12. A ship sails E. N. E. at the rate of 5 knots per hour, and is drifted by the tide E. by S. at the rate of 3 knots per hour. Find *her real motion* in magnitude. *Ans.* 7.68 knots per hour.

13. Find the direction of her real motion.

Ans. 80° 3′ E. of N.

14. In example 10, at what angle with the bank should the boat be rowed, in order to land at the opposite point of the river?

Ans. 62° 36′ against the stream.

4. **Laws of Motion.**—The laws of motion are simple statements of observed facts, from which can be deduced, without any further appeal to experiment, the laws of the most complicated motions. They are summed up by Sir Isaac Newton in three laws. These laws of motion are the following:—

LEX. I.

"*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*"—Princip. Math., Jes. Edit. tom. i. p. 15.

A body will continue in its state of rest, or state of uniform motion in a right line, unless compelled to alter that state by force impressed upon it.

This law expresses the fact, that matter is indifferent to rest or motion; or that a state of uniform rectilinear motion is as much a proof as a state of rest that a body is not acted upon by force; or rather, that the forces acting upon it equilibrate each other. One of the best illustrations of this fundamental property of matter is derived from the facts observed in driving a train of carriages on a horizontal railroad; it is found that it requires a constant expenditure of force, i. e. of coals or steam, to maintain a constant velocity. In this case the forces which tend to destroy the motion of the train are friction, and the resistance of the air, both of which are constant at a constant velocity; when these forces are equilibrated or destroyed by a constant expenditure of propelling force, the train is in the same condition as if it were acted upon by no forces whatever, and therefore,

in accordance with the first law of motion, continues to move uniformly forward until stopped by some external force. The property of matter which is described in this law of motion, viz., its incapacity to alter its condition, whether of rest or of motion, is frequently called the *inertia of matter.*

LEX II.

"*Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*"—Princip. Math., Jes. Edit. tom. i. p. 15.

The change in the quantity of motion produced is proportional to the force applied, and takes place in the direction of that force.

The kind of 'motive force impressed,' intended by Newton in this law, was primarily a blow or Momentum, and secondarily a Pressure, or ordinary statical force or weight. The effect of a blow is instantaneous; the velocity produced by it is its dynamical measure, and that velocity takes place in the direction of the blow or momentum; while the effect produced by a Pressure, or Statical force, takes time to develope itself, as will be fully shown in Chapter III. In either case, of a Momentum, or a Statical Pressure, this law of motion contains three distinct statements, which, for greater clearness, we shall give separately: the first relates to direction; the second and third to magnitude.

First Statement.—*The motion produced in a body by the application of a force takes place in the direction of the force, and is independent of any previous motion existing in the body.* The fact expressed by this proposition is generally called the *Law of the Composition of Velocities*, and has been fully illustrated in the preceding section.

Besides the statement relating to direction which has been just explained, the second law of motion contains the two following statements relating to magnitude.

Second Statement.—*The velocities produced in the*

same body by different statical forces acting for the same time, are proportional to those forces.

THIRD STATEMENT.—*If the same statical force be applied to move different bodies, the velocities produced in the same time will be inversely proportional to the quantities of matter in the bodies moved.*

Both these statements may be expressed algebraically in a single formula, as follows:—Let F represent the Statical force in pounds, ounces, &c.; and let f represent the Dynamical force, i. e. the velocity of the body acquired in the unit of time; and let m represent the Quantity of matter in the body moved.

The equation,

$$F = mf, \tag{2}$$

or its equivalent,

$$f = \frac{F}{m},$$

expresses that the Dynamical measure of force f, is directly proportional to the Statical measure of force F, and inversely proportional to the Quantity of matter m; or that *if the same Statical force be applied to move different bodies, it will produce in each of them the same Quantity of motion in the same time.*

The meaning of the second and third Statements just given may, perhaps, be rendered more clear by the following illustration:—

If a force equal to 1 lb. be applied to move a body which contains a quantity of matter represented by unity, it will produce a velocity of 1 foot per second in one second; if a force of 2 lbs. be applied, it will produce a velocity of 2 feet per second; if 3 lbs., 3 feet per second; and so on: this is the second Statement.

If a force of 1 lb. be applied to produce motion in a body whose quantity of matter is represented by unity, in one second it will produce a velocity of 1 foot per second; if it be applied to move a body containing twice

the quantity of matter, the velocity produced will be only $\frac{1}{2}$ ft. per second; if the quantity of matter be three times as great, the velocity will be $\frac{1}{3}$ ft. per second; and so on: this is the third Statement.

EXAMPLES.

In equation (2), f is understood to be expressed in feet; but the units to which F and m are referred, although arbitrary, are so connected, that one of the units being fixed, the other is also determined.

Let us suppose, for example, that we select for the unit of matter the quantity of matter contained in a cubic inch of distilled water at a temperature of 60° Fahrenheit, then the unit of weight to which F is referred may be thus found. The unit of F is the force which will produce in one second a velocity of one foot per second in a cubic inch of distilled water; but it is found from experiment, that if a cubic inch of water be allowed to fall in vacuo at Dublin, its weight, i. e. the force due to terrestrial gravity which acts upon it, will produce in one second a velocity of 32.1948 feet per second; hence,

$$\text{Unit of } F : \text{weight of cubic inch of water} :: 1 : 32.1498.$$

Or, finally—

$$\text{Unit of } F = \frac{\textit{weight of cubic inch of water}}{32.1948}.$$

If the weight of the cubic inch of water be expressed in grains, the unit of F will therefore be 7.842 grs.; because the weight of a cubic inch of water is 252.5 grs.; in other words, it would require a force of 7.842 grs. to produce in a cubic inch of distilled water at 60° F. a velocity of 1 ft. per second. Hence, if in equation (2) m be referred to cubic inches of water, F must be expressed in the units just found. If, on the other hand, the unit of F be given, the unit to which m is referred may be found as follows:—Let us suppose that F is expressed in pounds avoirdupois: from what has been already stated, it appears that a force of one pound would produce in one second a velocity of 32.1948 feet in the quantity of matter contained in a pound weight of water, hence it would produce a velocity of one foot per second in the quantity of matter contained in 32.1948 lbs. weight of water, i. e. in the quantity of matter contained in 892.5 cubic inches of distilled water at the temperature 60° F. (because the pound avoirdupois contains 7000 grains). The unit of volume to which m is referred is, *therefore, in this case,* 892.5 cubic inches.

1. If the unit of weight in equation (2) be one grain, find the corresponding unit of volume to which m is referred.

Let x denote the required unit of volume, expressed in cubic inches; then,

$$x \times 252.5 \text{ grs.} : 1 \text{ gr.} :: 32.1948 \text{ ft.} : 1 \text{ ft.}$$

Ans. 0.127 cubic in.

2. If the unit of weight be one ton, express the corresponding unit of m in cubic yards of water at 60° Fahrenheit.

Let x denote the required unit of volume, expressed in cubic feet; then, since a cubic foot of water is 62.32 lbs.,

$$x \times 62.32 \text{ lbs.} : 2240 \text{ lbs.} :: 32.1948 \text{ ft.} : 1 \text{ ft.}$$

Hence,

$$x = 1157.2 \text{ cub. ft.}$$

Ans. 42.858 cub. yds.

3. If the unit of matter be the quantity of matter contained in a cubic foot of water, find the corresponding unit of weight.

Let x denote the unit of weight in lbs.; then,

$$x : 62.32 \text{ lbs.} :: 1 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 1.935 lbs.

4. If the unit of matter be the quantity of matter contained in a cubic inch of lead (sp. gr. = 11.35); find the corresponding unit of weight.

Let x denote the unit of weight expressed in grains; then,

$$x : 11.35 \times 252.5 \text{ grs.} :: 1 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 89 grs.

5. If a force equal to 3 lbs. produce in one second a velocity of 0.317 feet in a given body; find the quantity of matter contained in the body.

Let x denote the number of cubic feet of water, which contain the same quantity of matter as that in the body; then,

$$x \times 62.32 \text{ lbs.} : 3 \text{ lbs.} :: 32.1948 \text{ ft.} : 0.317 \text{ ft.}$$

Ans. Quantity of matter contained in 4.883 cubic ft. of water.

6. If a force of 13.16 lbs. produce in a cubic foot of matter, in one second, a velocity of 4.16 ft., find the specific gravity of the matter.

Let x be the required specific gravity; then,

$$x \times 62.32 \text{ lbs.} : 13.16 \text{ lbs.} :: 32.1948 \text{ ft.} : 4.16 \text{ ft.}$$

Ans. 1.634.

7. If a velocity of 42.310 ft. be produced in one second, in a cubic inch of gold (sp. gr. = 19.35); find the magnitude of the force required to produce this velocity.

Let x denote the required force expressed in grains; then,

$$x : 19.35 \times 252.5 \text{ grs.} :: 42.31 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 6421 grs.

8. If a force of 17 lbs. produce a velocity of 14 feet in a cubic foot of matter, in one second, find its specific gravity.

Let x denote the required specific gravity; then

$$17 \text{ lbs.} : x \times 62.32 \text{ lbs.} :: 14 \text{ ft.} : 32.1948 \text{ ft.}$$

Ans. 0.627.

LEX III.

"*Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*"—Princip. Math., Jes. Edit. tom. i., p. 16.

Action and reaction are equal, and act in opposite directions: or if two bodies mutually act upon each other, the quantities of motion developed in each, in the same time, are equal and in opposite directions.

In this law of motion, by action and reaction we are to understand quantity of motion, or momentum; and since bodies in nature are observed to act mechanically upon each other, in one or other of two ways, viz., by percussion, or attraction and repulsion, the statement contained in this law of motion has two kinds of application, according as the forces considered are of the nature of percussions, or ordinary statical forces, such as attractions, which are measured by weights. We shall have occasion to apply this law of motion, which is of

the utmost value in practical mechanics, and founded on most accurate experiments, to the problem of the Collision of Bodies, which will be discussed in Chapter v.

The laws of motion which have been given in this chapter have been tested by the most careful and accurate experiments; but from their very nature they do not admit of any very direct mode of proof. The learner must be satisfied to take them for granted at the commencement, and he will find that, as he progresses in the study of Dynamics, the proofs of the truth of the fundamental laws will multiply; for since the whole science is based upon these laws, every complicated deduction from them, which is verified by experience, will afford a fresh proof of the truth of the laws themselves, and thus furnish evidence which, though *indirect*, is as certain as direct evidence.

As an illustration of this kind of proof, we may mention the prediction of the time and circumstances of an eclipse of the sun or moon; this prediction is a deduction from the elementary laws of motion, and its exact agreement with the observed facts of the eclipse is obviously a strong proof of the truth of those laws of motion from which the prediction is derived.

CHAPTER II.

ON THE WORK DONE BY AGENTS OR MACHINES MOVING UNIFORMLY.

1. Work done by a Force.—2. Constancy of Work done by a Force in a Machine moving uniformly.—3. The Lever moving uniformly.—4. The Wheel and Axle moving uniformly.—5. The Inclined Plane.—6. The Pulley.—7. The Gain in Power is Loss in Time.

WHEN a body is in a state of uniform motion, it appears from Newton's First Law of Motion, that the forces acting upon it are in equilibrium; but any attempt to stop the body, at once developes the force residing in it. When a living agent or a machine is doing work, in a state of uniform motion, the work applied must be equal to the work of resistance, for, since the motion is uniform, the forces acting must be in equilibrium with the resisting forces. We shall consider in this chapter several cases of this kind of work, which is of constant occurrence in practice.

1. Work done by a Force.—The *work done* by a force is expressed by the weight *lifted* through a given height by the force, and is therefore the product of a weight by a height—

$$\text{Work done} = \text{weight} \times \text{height}.$$

This equation is evidently true, because the work is proportional to the weight raised, and also to the height to which it is raised; if either of these be changed, the work done is changed in the same ratio, and is therefore proportional to the product of the weight and height to which it is raised.

Let W denote the work done, P the weight raised, and h the height to which it is raised, by the action of a force; then

$$W = Ph. \tag{3}$$

In this equation P is expressed always in pounds avoirdupois, and h in feet; the unit of work to which W is referred is sometimes called the "foot-pound," to signify one pound lifted through one foot.

If w denote the work done in one minute by an agent, such as a man, horse, &c., working in a particular manner, t the number of minutes the agent is employed, and n the number of agents at work, it is evident that the work done in the time t is

$$W = wnt. \tag{4}$$

Tables have been constructed, founded on experiment, showing the quantity of work per minute which various agents are capable of accomplishing: from these Tables the value of w is known for each agent; and by their aid many useful questions in practical mechanics may be solved.

If we wish to compare together the work done by different agents, we make use of the following equation:—

$$wnt = w'n't', \tag{5}$$

in which the accented letters refer to one set of agents, and the letters without accents to the other.

The following Table is taken from the results of Navier, Morin, and other writers:

TABLE *of Work done per Minute in Foot-pounds.*

	Duration of Labour.	Work.
MAN.	hours.	ft. lbs.
1. Raising his own weight on a ladder, ...	8	4222
2. Raising weights with a cord and pulley . . .	6	1559
3. Raising weights with the hand, ...	6	1472
4. Raising weights on the back of a ladder, . .	6	1126
5. Raising weights on a slope with a wheelbarrow,	10	519
6. Shovelling materials to a height of 5 ft., . . .	10	467
7. Working inside a treadmill—		
a. On a level with the axis, ...	8	3897
b. Near the bottom of the wheel (24°), . .	8	3637
8. Pushing or drawing horizontally, ...	8	3117
9. Pushing and drawing alternately, vertically, .	10	2380
10. Working a windlass, ...	8	2598
11. Working with the arms and legs, as in rowing,	8	4000
ANIMALS.		
12. Horse yoked to a waggon, walking, ...	10	27279
13. Horse yoked to a carriage, trotting, ...	4.5	41914
14. Horse yoked to a whim, walking, ...	8	17356
15. Horse yoked to a whim, trotting, ...	4.5	25980
16. Ox yoked to a whim, walking, ...	8	15588
17. Mule yoked to a whim, ...	8	11691
18. Ass yoked to a whim, ...	8	4849
19. Steam-horse of Watt and English writers, . .	24	33000
20. Steam-horse of French writers, ...	24	32475

In equation (3) the work done is expressed absolutely as the product of a weight and a height; in equation (4) it is expressed in terms of a number of agents of a given description, acting for a certain time; equating these two values of W, we find

$$Ph = wnt. \tag{6}$$

In this equation there are five quantities:—

P = the *weight* lifted,
h = the *height* through which it is lifted,
w = the *work* done by an agent per minute,
n = the *number* of agents,
t = the *time* during which the agents work;

any four of which being given, the fifth may be found. The following are a few of the numerous important practical questions which may be solved by equations (5) and (6):—

EXAMPLES.

1. What should be the horse power of a steam-engine capable of raising 750 tons of coal per day of 12 hours from a pit 100 fathoms deep?

Since the agent here employed is the steam-horse, we have—

$P = 750 \times 2240$ lbs,
$h = 6 \times 100$ ft.
$w = 33000$ ft. lbs.
n = required horse power.
$t = 60 \times 12$ min.

Ans. 42.4 H. P.

2. An engine of 20 horse-power is employed to pump water to a height of 60 ft. to supply a town of 5000 houses with water, at the rate of 80 gallons per house; how long must the engine work each day?

Here we have, since the gallon of water is 10 lbs.,

$P = 5000 \times 80 \times 10$ lbs.
$h = 60$ ft.
$w = 33000$ ft. lbs.
$n = 20$.
t = required time in minutes.

Ans. $t = 6^h. 4^m.$

3. A horse drawing a waggon at the rate of 2 miles per hour exerts a traction of 154 lbs., what is the work done per minute?

Here we have

$P = 154$ lbs.
$h = 5280 \times 2$ ft.
w = required work.
$n = 1$ horse.
$t = 60$ min. *Ans.* w = 27104 ft. lbs.

4. Five steam engines, of a total power of 1310 horses, were employed in pumping water from the shaft of Dawdon colliery, in Durham; what quantity of water could be raised by them from a depth of 73 fath. per hour?

Here we have

$P =$ required quantity.
$h = 73 \times 6$ ft.
$w = 33000$ ft. lbs.
$n = 1310.$
$t = 60$ m. *Ans.* $P. = 592191$ galls.

5. Four horses working a whim are required to raise water from a shaft at the rate of 3213 galls. per hour; what is the greatest depth to which the shaft can be unwatered by means of these horses?

Here

$P = 32130$ lbs.
$h =$ required depth.
$w = 17536$ ft. lbs.
$n = 4.$
$t = 60$ m. *Ans.* $h = 130.99$ ft.

6. A gang of 20 men is employed to pump water by means of a treadmill to a height of 40 ft.; in what time will they raise 10000 gallons, supposing one-third of the work applied to be lost by the friction of the pumps?

Here

$P = 10000 \times 10$ lbs.
$h = 40$ ft.
$w = 3897 \times \frac{2}{3}$ ft. lbs.
$n = 20$
$t =$ required time. *Ans.* $t = 1^{h}. 17^{m}.$

7. A party of 500 shovellers is employed to throw up an entrenchment in soft ground, and works for 10 hours per day; how many cubic yards of earth will be thrown up in 10 days, supposing each cubic yard to weigh two tons?

$P =$ required weight.
$h = 5$ ft.
$w = 467$ ft. lbs.
$n = 500.$
$t = 6000$ m.

Therefore, &c.

Ans. $= 62544$ cub. yds.

8. If 30 hodmen be employed on a building 30 feet high to carry bricks to the masons, how many bricks will they raise in 7 hours, *allowing 17 bricks to 125* lbs.?

Here

$P =$ required weight of bricks.
$h = 30$ ft.
$w = 1126$ ft. lbs.
$n = 30$.
$t = 420$ m. *Ans.* 64317.

9. Two men are employed to work a windlass, to raise materials from a shaft; they work for 8 hours per day for 10 days; in what time would four men working for 6 hours per day raise the same quantity of materials by ladders from the shaft?

Here we must use equation (5):—

$w = 2598$ ft. lbs.
$n = 2$.
$t = 10 \times 8 \times 60$ min.
$w' = 1126$ ft. lbs.
$n' = 4$.
$t' =$ required time.

Ans. $15^d. 2^h. 17^m.$

10. A stone quarry is unwatered by the labour of 24 oxen, working 8 at a time on two whims for 8 hours per day; what should be the horse-power of a pumping engine which could do the same work?

Here.

$w = 15588$ ft. lbs.
$n = 8$.
$t = 24 \times 60$ min.
$w' = 33000$.
$n' =$ required power of engine.
$t' = 24 \times 60$ min.

Ans. 3.77 H. P.

2. Constancy of work done by a force in a Machine moving uniformly.—In the third chapter of the Statics we have considered machines in a state of equilibrium, but as they are generally employed in a state of uniform motion, it is necessary to consider them in this point of view. When a machine passes from a state of rest into a state of uniform motion, the Power and Resistance each move through a certain space in a given time, and the *work done* by them is measured by their pressure in pounds multiplied into the space in feet through which they move, in the direction of the

force, in one minute. The principle according to which the equilibrium of machines in a state of uniform motion is determined, is called the principle of the constancy of work done, and is explained in the following paragraph.

A force may be transmitted from one point of application to another by means of various instruments, such as cords, bars, wheels, &c.; but it is not possible for these inanimate objects to generate work, and whatever diminution the work done by the force may undergo by friction, we are certain that it can never be increased. We always find, in fact, that the *work applied* to any machine is greater than the *work done* by the machine, and are led by numerous experiments and observations to conclude that this loss of work is due to friction; so that if it were possible to construct a machine without friction, the work applied would be equal to the work done, and the celebrated problem of "perpetual motion" would be mechanically possible. If we imagine a machine moving uniformly without friction, the work done by any part of the machine is constant and equal to the work applied to move the machine. This is called the "principle of constancy of work done," and is applicable to machines moving uniformly without friction.

The principle of constancy of work may be thus stated:—

Let P be the power acting at any part of a machine;

Let p be the distance through which it moves in one minute in its own direction;

Let R be the resistance acting at some other part of the machine;

Let r be the distance through which it moves in one minute; then

$$Pp = Rr. \qquad (7)$$

In this equation Pp and Rr are the work done by the power and resistance, and they are equal to each other *when the machine* moves uniformly without friction.

3. **The Lever.**—If a lever turn uniformly round its fulcrum F; at the end of any time, the power and resistance will have passed through arcs of circles, whose lengths are proportional to their radii FP and FR. Substituting these arcs or the radii FP and FR for p and r in equation (7), we find

$$\text{Power} \times \text{PF} = \text{Resistance} \times \text{RF};$$

or

$$\text{Power} : \text{Resistance} :: \text{RF} : \text{PF};$$

i. e. *the power is to the resistance inversely as the arms of the lever.*—Q. E. D.

4. **The Wheel and Axle.**—If the wheel and axle revolve uniformly, in a single revolution, the power passes through a space p, which is the circumference of the wheel, and the resistance, in the same time, passes through a space r, which is the circumference of the axle. Substituting these values for p and r in equation (7), we find,

Power × circumference of wheel = Resistance × circumference of axle;

Or, since the circumferences of circles are proportional to their radii,

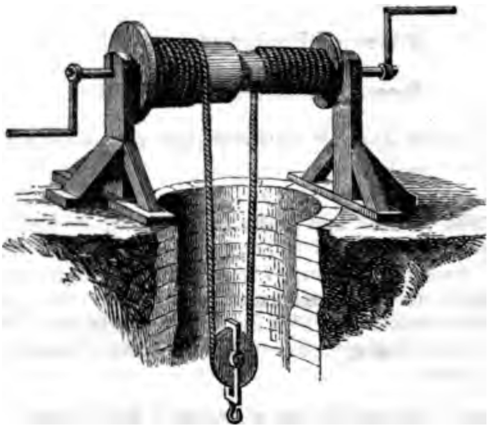
The power is to the resistance as the radius of the axle is to the radius of the wheel.—Q. E. D.

The wheel and axle is sometimes modified, as shown in the following figure. In this case the work done by the Power in a single revolution is the Power multiplied by the circumference described by the winch-handle; and since, in a single revolution, the rope is diminished by a length equal to the difference of the circumferences of the axles, the weight will be lifted through *half* this difference, and therefore, by the prin-

ciple of constancy of work done by a machine, we have

$$P \times C = W \times \frac{c - c'}{2};$$

where C is the circumference of the circle described by



the winch-handle, and c, c' are the circumferences of the axles. Hence we find

The power is to the weight as the difference of the circumferences, or radii of the axles is to twice the circumference, or radius of the circle described by the winch-handle.

5. **The Inclined Plane.**—The only case of the inclined plane which is of any practical importance is that in which the power is applied parallel to the plane, as in the case of drawing loads up hills, &c. In this case, while the power moves through the length of the plane, the weight is lifted through the height of the plane, and therefore, by equation (7),

Power × length of plane = Resistance × height of plane,

Or,

The power is to the resistance as the height of the plane is to its length —Q. E. D.

6. **The Screw.**—In the case of the screw it is evident that while the power describes the circumference of a circle, the weight or resistance is moved through a distance equal to the interval between the threads of the screw, and consequently, by equation (7),

Power × circumference of circle through which it moves = Resistance × interval between the threads.

Or,

The power is to the resistance as the interval between the threads is to the circumference of the circle described by the power.—Q. E. D.

7. **The Pulley.**—Whatever be the complexity of the system of pulleys considered, the principle of constancy of work done enables us to determine the ratio of the power to the resistance; for we have simply to multiply the power by the space through which it passes, and to equate this product to the resistance multiplied by the space through which it is lifted in the same time.

Thus, in the single moveable pulley, the power P will evidently descend through a space twice as great as that through which the resistance R is lifted, and, therefore, the power is half the resistance.

In the system (p. 56) the power will descend through six times the distance through which the weight R is raised, and, therefore, the power is only one-sixth of the weight lifted; and, in a similar manner, the ratio of the power to the weight lifted may be found in other cases.

8. **The Gain in Power is Loss in Time.**—The prin-

ciple of the constancy of work done by a force, which has been illustrated by the cases of the common mechanical powers, is of universal application (always neglecting friction), and may be used to find the ratio of the power to the resistance in the most complicated as well as in the simplest machines; and it possesses this practical advantage over all other methods of investigation, that it only requires us to know the spaces traversed by any two points of the machine in the same time, to be able to state the ratio of the forces acting at these points. The spaces described by the different parts of a machine are found by direct experiment, and are easily measured; and hence this principle, although under a different name, has been long familiar to practical mechanicians. It is generally stated in the following form, that "what is gained in power is lost in time;" or, in other words, that all parts of any machine do the same amount of *work*, because, if any point moves with a greater force than another, it must move slower, in such a manner as that the product of the force into the space it moves through in a given time shall be the same for this point as for all other points of the machine. In every case of machinery, however complicated, the equation

$$Pp = Rr, \tag{7}$$

is always true.

Exercises on the Work done by Agents moving uniformly.

1. Coulomb observed that in the ordinary operation of pile driving, each man lifts 19 kilogrammes of the ram; at every pull the ram is raised 1.1 metre; the men make 20 pulls in one minute, and after three minutes of exertion, rest for as long a time, and then begin again. Thus, their daily labour lasting 6 hours, the duration of fatiguing exertion does not exceed 3 hours. Find the work done per day. (*Venturoli.*) *Ans.* 75,240 kilogram-metres, or 242 foot-tons.

2. Lamandé, architect of the bridge of Jena, found that 38 labourers, working 10 hours a day, made in each hour 12 efforts, each of 30 pulls; the ram weighed 587 kilos., and was raised through 1.45 met. each stroke. Find the diurnal work of man. (*Venturoli.*)
Ans. 80,635 kilogr.-metres, or 260 foot-tons.

3. The following observation is recorded by Coulomb, made on men employed in turning a lever:—Under a continued labour they each exerted a force equivalent to 7 kilos., and made 20 turns in a minute, the circumference of the turn being 2.3 metres; they continued at work 8 hours, but owing to the rests of which they have need from time to time, the duration of the daily fatigue does not exceed 6 hours. Find the diurnal work of man. (*Venturoli.*)
Ans. 115,920 kilogram-metres, or 374 foot-tons.

4. Porters were employed to carry goods to a distance of 2000 metres, and each was found to carry in the course of the day 348 kilogrammes, at six journeys, loaded with 58 kilogrammes at a time. If each porter weigh, on an average, 55 kilos., find the daily work expressed in kilogrammes, drawn horizontally along a road. (*Venturoli*). *Ans.* 2,016,000 kilogram-metres.

5. On questioning a number of pedlars, who travel loaded with their packs, Coulomb found that they stated that, with a load of 44 kilos., they travelled 19000 metres a day. Assuming their average weight to be 55 kilos., find the daily work in horizontal kilogram-metres. (*Venturoli.*) *Ans.* 1,881,000 kilogram-metres.

Note.—Coulomb, from whom we have partly taken examples 4 and 5, solves them without taking any account of the weight of the porters or pedlars, and thus obtains for answers; Porters = 696,000 horizontal kilogram-metres, and Pedlars = 836,000 k. m. hor.; this latter result being greater than the former. Coulomb explains it by adding, that he suspects some exaggeration in the assertion of the Pedlars: it appears, on the contrary, that when proper account is taken of the weight of the labourers in estimating the work done by them, that the pedlars actually do less work than the porters, and that Coulomb unintentionally did them an injustice.

6. From many considerations we are led to believe that a man, in good condition, and resting on Sundays, as a labourer does, can walk 20 miles per day, as a continued labouring effort. Assuming his weight at 55 kilogrammes, find the work done in horizontal kilogram-metres. *Ans.* 1,760,000 kilogram-metres.

7. Assuming the results of examples 1, 2, 3, to determine the daily work of man in lifting weights; and the results of 4, 5, 6, to determine his ability to carry weights along a horizontal road; find the coefficient of traction of a man walking along such a road.

Ans. $\frac{1}{20.81}$.

8. A number of porters, according to Coulomb, carried wood, mounting a convenient staircase, 12 metres high, and its height one-third of its horizontal base; they made 66 journeys per day, and carried each 68 kilogrammes of wood on each journey. Find the daily work done. (*Venturoli.*)

In solving this question, Coulomb takes no account of the weight of the labourers, except in a general way; if we assume, as before, the average weight of each labourer to be 55 kilos., we shall have the work done by finding the weight lifted per day, and adding to it $\frac{1}{20}$th part of the weight drawn horizontally; therefore,

Ans. 112,028 kilogrammes lifted through one metre per day.

Or, if we take account of the labourers returning unloaded along the base of the staircase.

Ans. 118,562 kilogram-metres, or 381 foot-tons.

9. In the conveyance of earth by wheelbarrows, the weight of the barrow is 30 kilos.; it is loaded with 70 kilos.; the weight supported by the labourer's hand is from 18 to 20 kilos., the remainder being supported by the ground—at the same time he pushes with a force which may be estimated at from 2 to 3 kilos.; he makes 500 journeys per day of 29.226 metres each, returning with the barrow empty. (*Venturoli.*)

This question involves a mixed action, and as we venture to differ from Coulomb and Venturoli (who differ from each other), in their solution of it, we shall place before our readers all three solutions.

a. Coulomb's Solution.—Let the weight 70 kilos. be multiplied by the distance travelled. $29.226 \times 500 = 14613$ metres.

Ans. 1,022,910 kilogram-metres.

b. Venturoli's Solution.—To measure the effect of the force of the labourer in the 14613 metres, we ought to multiply by 22 kilos., which is about the sum of the *lift* and *push* of the labourer's hand; this gives 321,486; and since, in returning, he describes the same space, exerting a force of about $5\frac{1}{2}$ kilos., this part of the exertion will be $14{,}613 \times 5.5 = 80{,}371.5$ kilogram-metres, and finally the total work. *Ans.* 401,857.5 kilogram-metres.

c. Our own Solution.—The weight of the labourer is omitted altogether from both the preceding solutions; and that of the wheelbarrow from that of Coulomb; but independently of this, we cannot agree with either of them; the problem of a man pushing a wheelbarrow appears to be the same as that of a horse drawirg a waggon laden with sacks of goods, from which some of the sacks are taken, and placed on the horse's back. If the coefficient of traction of the horse himself and of the waggon along the road were equal, it would be a matter of indif-

ference how much of the load were placed on his back, and how much on the waggon, and the principle of Coulomb's solution would be correct; the traction of a horse or man along a road is $\frac{1}{20}$th, while that of the waggon or wheelbarrow varies from $\frac{1}{30}$th to $\frac{1}{40}$th, according to the state of the road and waggon.

According to our view of the subject, therefore, the correct solution of the problem is the following:—

Let the weight of the labourer be 55 kilos.;
The coeff. of traction of the labourer is $\frac{1}{20}$th;
That of the loaded wheelbarrow is given 2.5 kilos. in 100, or $\frac{1}{40}$th;
One-fifth of the weight is lifted by the hand;
Therefore the direct journeys require a work of

$$14613\left[\frac{20+55}{20}+\frac{80}{40}\right] \text{ kilogram-metres,}$$

and the return journies require a work of

$$14613\left[\frac{6+55}{20}+\frac{24}{40}\right] \text{ kilogram-metres,}$$

or, adding together,

Ans. 137,362 kilogram-metres,
Or 441 foot-tons.

10. The paviours of Dublin work under the following conditions:—

Weight of rammer, ...	5 st. 9 lbs.
No. of blows, ...	78 in $2^m. 45^s$.
Rest, ...	$3^m. 30^s$.
Height lifted, ...	16 inches.
Hours of labour, ...	10 hours.

Calculate from these data the work done per day by each man?

Ans. 352 foot-tons.

11. The prisoners confined in the military prison of Dublin perform their punishment shot-drill under the following conditions:—

"Each man lifts a 32 lb. shot to his breast (3 ft.) from a tressel, carries it through 9 ft. (4 paces by drill), and lays it down on a similar support; he then returns unloaded, and takes up another shot, and so repeats the double journey; of course, after a certain time, all the 32 lb. shots are transferred from one side to the other of the working gang, and they must then reverse the order of proceedings, and carry back the shots; six of the double journies occupy one minute. If we assume that the same work is done in laying down as

in lifting up the shot, prove the following expression for the work done per minute :—

$$\text{Work} = \left(\frac{(2w + 32)a}{20 \times 2240} + \frac{32 \times 2\,h}{2240} \right) \times n,$$

where

w = weight of man in lbs.
a = distance in feet to which the 32 lb. shot is carried.
h = height in feet to which it is lifted.
n = number of double journeys per minute.

Assuming the average weight of the prisoners at 141 lbs, and that they are employed for 3 hours, find the work done?

Ans. 161 foot-tons.

12. A coal pit, 1200 ft. deep, is flooded by a feeder, discharging 900 gallons per minute at the bottom; if the united pumping engines of the pit be 400 h. p., and are in such bad condition, that they only utilize 60 per cent. of their nominal horse-power; it is required to find whether they are capable of keeping the pit in fork?

Ans. The water will rise in the pit 320 ft.

CHAPTER III.

ON RECTILINEAR MOTION AND CONSTANT FORCE.

1. Relation between Velocity and Time. — 2. Relation between Space and Time.—3. Relation between Velocity and Space.—4. Motion of Falling Bodies.—5. Motion of Bodies on Inclined Planes. — 6. Experimental Proofs of the Laws of Rectilinear Motion.

1. **Relation between Velocity and Time.**—We shall, in this chapter, investigate the laws of motion of bodies acted upon by constant forces, and moving in right lines. The dynamical force is supposed to be known, and the body is supposed to set out from a state of rest. The first thing to be done is to determine the velocity produced in a given time.

Let f denote the dynamical measure of the force, i. e. the velocity acquired at the end of one second, and let v represent the velocity at the end of t seconds. Then,

$$v : f :: t : 1,$$

i. e. the velocity acquired in t seconds is to the velocity acquired in one second as $t : 1$; hence,

$$v = ft. \tag{8}$$

This equation expresses the relation between the velocity, v, the dynamical force, f, and the time, t, during which it acts; and enables us, from any two of these quantities, to calculate the third.

EXAMPLES.

1. A velocity of 14 feet per second is produced by a force acting for 3 minutes. Calculate the force. *Ans.* 0.0777 ft. per second.

2. A force acting for 13 seconds produces a velocity of 4 miles per hour. Find its magnitude *Ans.* 0.451 ft. per second.

3. A force equal to 7 feet per second acts for 3 minutes. Find the velocity produced. *Ans.* 1260 ft. per second.

4. A force equal to 32.195 feet per second acts for 5 seconds. Find the velocity produced. *Ans.* 160.975 ft. per second.

5. Find the time during which a force of 11.16 feet must act to produce a velocity of 100 feet per second. *Ans.* 8.96 seconds.

6. During what time must a force of 32.195 feet act to produce a velocity of 375 feet per second? *Ans.* 11.64 seconds.

2. **Relation between Space and Time.**—Let us suppose the velocity of the body acted on by the constant force to increase *per saltum*, and to receive equal increments of velocities in equal intervals of time. If the final velocity be v, and the whole time t be divided into n equal intervals, it is evident that the velocity during the first interval is $\frac{v}{n}$, during the second, $\frac{2v}{n}$, during the third, $\frac{3v}{n}$, and so on. And since the space described by a body moving uniformly is equal to the product of the velocity and time (1), it follows that

$$\text{Space described in first interval} = \frac{v}{n} \times \frac{t}{n} = \frac{vt}{n^2},$$

$$\text{,, second ,,} = \frac{2v}{n} \times \frac{t}{n} = \frac{2vt}{n^2},$$

$$\text{,, third ,,} = \frac{3v}{n} \times \frac{t}{n} = \frac{3vt}{n^2},$$

$$\text{,, } n^{th} \text{ ,,} = \frac{nv}{n} \times \frac{t}{n} = \frac{nvt}{n^2}.$$

If s be the whole space described, it must be equal to the sum of the spaces described in the successive intervals, and therefore, adding, we find

$$s = \frac{vt}{n^2} \times \{1 + 2 + 3 + 4 + \&c. + n\};$$

but the series included within the parenthesis is equal to $\frac{n(n+1)}{2}$ (*vid.* Manual of Algebra); therefore

$$s = \frac{vt}{2} \times \frac{n(n+1)}{n^2} = \frac{vt}{2}\left(1 + \frac{1}{n}\right).$$

If now we suppose the velocity to increase, not *per saltum*, but *continually* and in proportion to the time, we must suppose the intervals of time to be indefinitely small, i. e. the number of intervals n indefinitely great; on which supposition $\frac{1}{n} = 0$, and therefore

$$s = \frac{vt}{2}.$$

Now, if a body be acted on by a constant force, the velocity increases in proportion to the time, equation (8), and therefore, *If a body move from rest under the action of a constant force, the space described is equal to half the product of the time and the final velocity.*

In this case, as $v = ft$, by equation (8),

$$s = \frac{f}{2}. \qquad (9)$$

This equation contains three quantities, viz., s the space, t the time, and f the dynamical force. Given any two of these, we can calculate the third.

EXAMPLES.

1. If a body, under the influence of a constant force, move through 5 feet in 3 seconds, find the magnitude of the force which animates it. *Ans.* 1.111 ft. per sec.

2. If the space described in 10 seconds be 300 yards, find magnitude of force. *Ans.* 18 ft. per sec.

3. If a body be acted on by a force of 11 feet, during 3 minutes, find the space described in that time. Ans. 178,200 ft.

4. If the force be 32.19 feet, and the time 7 seconds, find space described. *Ans.* 788.655 ft.

5. Find the time occupied by a body, acted on by a force of 12.17 feet, in describing 137 yards. *Ans.* 8.21 secs.

6. A body is moved by a force of 32.18 feet, which acts during 11 seconds. Find space described. *Ans.* 1946.89 ft.

3. **Relation between Velocity and Space.**—The velocity acquired by a body, acted on by a constant force, in describing a given space, may be found from equations (8) and (9), as follows:—

Squaring equation (8), we find

$$v^2 = f^2 t^2;$$

multiplying equation (9) by $2f$, we find

$$2fs = f^2 t^2;$$

hence,

$$v^2 = 2fs. \tag{10}$$

This equation contains three quantities, viz., f the dynamical force, s the space described, and v the velocity acquired. Any two of these being given, we can calculate the third.

EXAMPLES.

1. Find the force which would cause a body to acquire a velocity of 35 feet per second, after describing 100 yards. *Ans.* 2.04 ft. per sec.

2. If the space described be one mile, and the velocity acquired be 1142 yards per minute, find force. *Ans.* 0.309 ft. per sec.

3. A body acted on by a force of 13 feet per second describes 1100 feet. Find its velocity. *Ans.* 169.1 ft. per sec.

4. If the force be 32.190 feet, and the space described be 467 feet, find velocity acquired. *Ans.* 173.4 ft per sec.

5. With the same force, let the velocity acquired be 75 feet per second. Find space described. *Ans.* 87.3 ft.

6. If a body, animated by a force of 314 yards per minute, acquire a velocity of 411 feet per second; find space described. *Ans.* 5379.6 ft.

4. **Motion of Falling Bodies.**—The most important of all the forces with which we are acquainted is the force of Gravity, which, considered as a Dynamical force, is the cause of the motion of falling bodies. This force, as has been mentioned already, in the chapter on the Centre of Gravity, varies from place to place on the surface of the globe. The Table which is given in page 104 contains the values of the Dynamical force of gravity in English feet and decimals, reduced to the sea level, at some of the most important places on the earth's surface. It has been calculated from pendulum experiments in a manner which will be described in the chapter on the Pendulum. It is easy to see from it, that *at all places on the surface of the earth, the velocity acquired in one second by a body falling in vacuo is somewhat greater than 32 feet per second.*

The force of gravity is so important that it is usual in works on Mechanics to designate it by a special letter g, which in future we shall always use to denote the Dynamical force of gravity, expressed in feet.

If W denote the weight of a body, and m its mass, equation (2), when applied to the force of gravity, will become

$$W = mg. \tag{11}$$

This equation expresses the fact, that the weight of a body at different places on the earth's surface is proportional to the velocity acquired by it in falling for one second, and is an instance of the truth of the second law of motion.

Equations (8), (9), (10), referred to the force of gravity, become

$$v = gt. \tag{12}$$

$$s = \tfrac{1}{2}gt^2. \tag{13}$$

$$v^2 = 2gs. \tag{14}$$

These three equations contain the complete solution of all problems relating to falling bodies.

If the time be made equal to one second in equation (12), we find

$$v = g;$$

i. e. the velocity acquired in one second is the measure of the force of gravity.

If in equation (13), we make $t = 1$, we obtain

$$s = \tfrac{1}{2}g,$$

i. e. *the space described by a falling body in one second is half the velocity acquired in one second, or equal about* 16 *feet.*

If we now take t, in equation (13), successively equal to 2, 3, 4, &c., seconds, we find

$$s = \tfrac{1}{2}g \times 4 = 16 \times 4 \text{ ft.} = 64 \text{ ft. in 2 seconds.}$$
$$s = \tfrac{1}{2}g \times 9 = 16 \times 9 \text{ ft.} = 144 \text{ ft. in 3 ,,}$$
$$s = \tfrac{1}{2}g \times 16 = 16 \times 16 \text{ ft.} = 256 \text{ ft. in 4 ,,}$$

i. e. *the spaces described from the commencement of the motion vary as the squares of the number of seconds.*

If we subtract the space described in one second from the space described in two seconds, i. e. 16 ft. from 64 ft., we shall obtain the space described in the second second, or 48 ft.; and in like manner, if we subtract the space described in two seconds from the space described in three seconds, we should find the space described in the third second, and so on.

In general, the space described in $(n - 1)$ seconds is $16 (n - 1)^2$, and the space described in n seconds is $16n^2$; the difference of these numbers is the space described in the n^{th} second, or

$$\text{Space in } n^{th} \text{ second} = 16 \{n^2 - (n - 1)^2\} = 16 (2n - 1).$$

Hence, since $2n - 1$ is an odd number, *the spaces described by a falling body in successive seconds are proportional to the series of odd numbers.*

EXAMPLES.

1. Find the space described in the *ninth* second by a falling body. *Ans.* 272 ft.

2. Find the space described in *nine* seconds by a falling body. *Ans.* 1296 ft.

3. Find the space described between the *fourth* and *seventh* seconds by a falling body. *Ans.* 528 ft.

4. Find the space described between the *third* and *eleventh* seconds. *Ans.* 1792 ft.

Equation (13) enables us to calculate the height of any place approximately, by counting the time taken by a body (such as a stone) to fall from the top to the bottom. The simplest rule for this rough calculation is the following:—

The height of any place in feet is equal to the square of the number of quarter seconds occupied by a body in falling from the top to the bottom.

Since many watches are constructed to beat quarter seconds, the observation may be made by the ear, and the height will be given by the square of the number of beats of the watch during the descent of the body.

This rule is proved as follows:—The space described is equal, by equation (13), to $\frac{1}{2}g$, or 16 ft., multiplied by the number of seconds squared, or by the square of the quarter seconds divided by the square of 4: hence the 16 destroys in the numerator and denominator; leaving the height equal to the square of the number of quarter seconds.

Equations (12), (13), (14) contain the three quantities, space, time, and velocity; any two of which being given, the third may be calculated, provided the force of gravity be known at the given place.

TABLE OF THE DYNAMICAL FORCE OF GRAVITY.

Station.	Latitude.	Force of Gravity at Sea Level.	Names of Observers.
		FEET.	
1. London, ...	51° 31′ 8″ *N.*	32.1908	Sabine.
2. Paris, ...	48 50 14 *N.*	32.1820	Borda.
3. Dunkerque, . . .	51 2 10 *N.*	32.1895	Biot, Mathieu.
4. Bordeaux, . . .	44 50 26 *N.*	32.1691	Biot, Mathieu.
5. Leith, ...	55 58 41 *N.*	32.2040	Kater.
6. Rio de Janeiro, .	22 55 13 *S.*	32.1121	Freycinet.
7. Cape of Good Hope,	33 55 15 *S.*	32.1403	Freycinet.
8. Isle of France, . .	20 9 19 *S.*	32.1147	Duperrey.
9. Port Jackson, . .	33 51 39 *S.*	32.1412	Duperrey.
10. Isle S. Thomas, .	0 24 41 *N.*	32.0930	Sabine.
11. Isle Rawak, . .	0 1 34 *S*	32.0881	Freycinet.
12. Ascension, . . .	7 55 48 *S.*	32.0959	Sabine.
13. Sierra Leone, . .	8 29 28 *S.*	32.0927	Sabine.
14. Trinidad, . . .	10 38 56 *N.*	32.0917	Sabine.
15. Jamaica, ...	17 56 7 *N.*	32.1050	Sabine.
16. New York, . . .	40 42 43 *N.*	32.1600	Sabine.
17. Greenland, . . .	74 32 19 *N.*	32.2435	Sabine.
18. Spitzbergen, . .	79 49 58 *N.*	32.2526	Sabine.
19. Unst (Shetland), .	60 45 25 *N.*	32.2173	Biot.
20. Toulon, ...	43 7 9 *N.*	32.1668	Duperrey.

EXAMPLES.

1. Calculate the velocity acquired in 11ˢ by a body falling in Ascension Island. This problem is solved by equation (12); by the Table, $g = 32.0959$ ft., which multiplied by 11 is equal to the required velocity. 353.0549 ft. per sec.

2. Calculate the space described in 15ˢ by a body falling in Jamaica. By equation (13) the space is equal to half the force of gravity multiplied by the square of the time; but by the Table, $g = 32.1050$ ft., hence $\frac{1}{2}g = 16.0525$, which multiplied by 225 is equal to 3611.8125 ft.

3. Calculate the velocity acquired by a body in falling through 100 ft. in Spitzbergen. By equation (14) the square of the velocity is equal to the force of gravity multiplied by twice the space, i. e. 200 ft. multiplied by $g = 32.2526$, equal 6450.52; extracting the square root we find the velocity to be 80.315 ft. per sec.

4. Calculate the space described by a falling body at Port Jackson, in $3^s.151$. *Ans.* 159.5617 ft.

5. Calculate the time required to fall 316 ft. in New York. *Ans.* 4.43 secs.

6. Find the velocity acquired by a falling body in Paris, in 9 seconds. *Ans.* 289.638 ft. per sec.

7. If a body falls in Sierra Leone through 216.171 ft.; find the velocity acquired. *Ans.* 117.79 ft. per sec.

8. Find, in the same case, the time required to describe this space. *Ans.* 3.67 secs.

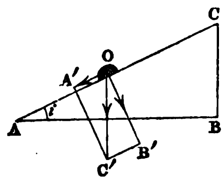
9. If a body falling at the Cape of Good Hope acquire a velocity of 71.30 ft.; find the space described. *Ans.* 79.085 ft.

10. Find also the time of describing this space. *Ans.* 2.21 secs.

11. Find the time required by a falling body in Bordeaux to acquire a velocity of 131.17 ft. *Ans.* 4.077 secs.

12. Find the time required by a body to fall 200 ft. in Trinidad. *Ans.* 3.530 secs.

5. **Motion of Bodies on Inclined Planes.**—If a body, instead of falling in a vertical line, be allowed to run down an inclined plane, the force which moves it may be varied at pleasure, by varying the inclination of the plane to the horizon. The magnitude of the force which impels a body down an inclined plane may be thus found :—Let AC be an inclined plane, whose inclination BAC is denoted by the letter i; let O be a body placed upon this plane; draw the vertical line OC′ equal to g, the gravity at the place, and complete the parallelogram C′A′OB′; OA′ is the component of gravity which moves the body down the plane; we shall call it f; OA′ is equal to OC′ multiplied by the sine of OC′A′, which is equal to BAC or i, because the triangles OC′A′ and BAC are similar; hence—

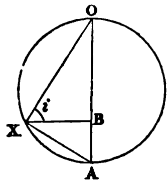


$$f = g \sin i. \quad (15)$$

This equation enables us, by knowing the amount of gravity, and the inclination of the plane, to solve all problems respecting the motion of bodies on inclined planes by means of equations (8), (9), (10).

There are two general properties of the motion of bodies on inclined planes which it is necessary to prove:—

1st. *The velocity acquired by a body in running down any inclined plane is equal to the velocity acquired in falling down the height of the plane.*



2nd. *If a circle be drawn in a vertical plane, and from its highest point* O, *chords be drawn; the time occupied by a body in running down any chord is constant.*

The first of these propositions is thus proved:—Let h represent the height of the plane, i. e. the line BC (fig. p. 105); by equation (10), $v^2 = 2fs$; substituting for f its value $g \sin i$ (15), we find $v^2 = 2gs \sin i$; but $s \sin i$ is equal to h, since s is equal to AC; therefore, finally, $v^2 = 2gh$

Or $$v = \sqrt{2gh}. \qquad (16)$$

This is the velocity acquired by a body running down from C to A; but the velocity acquired in falling from C to B under the influence of gravity is also equal to $\sqrt{2gh}$ by equation (14); consequently the velocity acquired in running down any inclined plane is equal to the velocity acquired in falling down its height.—Q. E. D.

The second proposition may be proved as follows:—The time of running down any chord OX is, by equation (9),

$$t = \sqrt{\frac{2s}{f}},$$

where s is equal to OX and $f = g \sin i$; i being the angle OXB, which is equal to OAX, because OAX is a right-angled triangle, and XB is perpendicular to OA; if we represent the diameter OA by h, we shall have OX equal to OA multiplied by the sine of OAX; or, $s = h \sin i$; substituting in the equation already found these values of f and s, we obtain, finally,—

$$t = \sqrt{\frac{2h}{g}}. \tag{17}$$

This is the time of running down the chord OX; but by equation (13) the time of falling down the diameter OA is also represented by $\sqrt{\frac{2h}{g}}$; hence the time of running down any chord is equal to the time of falling down the diameter, and is therefore constant.—Q. E. D.

EXAMPLES.

1. Find the time occupied by a body in running down 1000 ft. on an inclined plane at Dunkerque, the inclination of the plane being one degree. The time of descent, by equations (9) and (15), is $\sqrt{\frac{2s}{g \sin i}}$ which is equal to $\sqrt{\frac{2000}{0.5617}}$, because $s = 1000$, $g = 32.1895$, $\sin i = 0.01745$; extracting the square root, we find

Ans. $t = 59.67$ secs.

2. If the inclination of a plane at Toulon be 36′, find the velocity acquired in running down 1000 feet. By equations (10) and (15), $v = \sqrt{2gs \sin i}$, $= \sqrt{673.57}$, since $g = 32.1668$, $s = 1000$, and $\sin i = 0.01047$; extracting the square root, we find

Ans. $v = 25.95$ ft. per sec.

3. Find the velocity acquired in 10^s by a body running down an inclined plane in the Island of Ascension, whose inclination is 17′. By equations (8) and (15) we find $v = gt \sin i$, which, since $g = 32.0959$, $t = 10$, and $\sin i = 0.000494$, is equal to

Ans. 1.585 ft. per sec.

4. An inclined plane at Bordeaux has an inclination of 1 foot in 65; find the velocity acquired by a body running down it through a space of 427 ft. *Ans.* 20.558 ft. per sec.

5. Find the time of describing 427 ft. *Ans.* 41.54 secs.

6. Calculate the space described in 3^s. *Ans.* 2.227 ft.

7. Find the velocity acquired by a railway train in running down a gradient of 2164 ft., having a total fall of 31 ft.; the force of gravity being 32.19, and the resistance from friction and the air being estimated at 7 lbs. per ton. If f denote the force down the plane, and g the force of gravity,

$$f = g\left\{\frac{31}{2164} - \frac{7}{2240}\right\} = 0.36054 \text{ ft. per sec.};$$

and therefore *Ans.* 39.50 ft. per sec.

8. Find, under the same circumstances, the time of running down 1000 ft. *Ans.* 74.48 secs.

9. If the inclination of a plane in Greenland be 14°; find the space described in 5 secs. *Ans.* 97.503 ft.

10. Find the space described in 5^s, on a similar plane, in the Island of St. Thomas, Gulf of Guinea. *Ans.* 97.048 ft.

11. Calculate the velocity acquired in Ex. (9), in running down 100 ft. *Ans.* 39.5 ft. per sec.

12. Also the velocity acquired in 11^s. *Ans.* 85.80 ft. per sec.

6. **Experimental Proofs of the Laws of Rectilinear Motion.**—The laws of rectilinear motion of a body, under the influence of a constant force, have been deduced from the laws of motion laid down in the first chapter; but this method, although the most systematic, is not the most natural, since the Laws of Motion were originally deduced from observations made on the motion of bodies acted on by a constant force. The only constant force suitable for such observations, with which we are acquainted, is the force of gravity; but as the velocity produced by this force is too great for direct experiments, it is necessary to devise some method by which

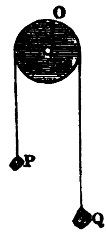
the force of gravity shall be so diminished, and the motions produced by it rendered so slow, as to render the observation of them possible. This object has been effected in two ways: one of which is the method employed by Galileo, in his experiments on falling bodies; and the other is the celebrated method used by Mr. Atwood for establishing the elementary laws of Dynamics.

The *Method of Galileo* consists in the use of an inclined plane, or rather inclined cord, about thirty feet in length, stretched between two points, one of which is higher than the other, on which a weight is made to traverse by means of a small pulley.

It appears from equation (15), that the force which actuates the body is gravity, multiplied by the sine of the angle of inclination: consequently, by increasing or diminishing the inclination, we can increase and diminish the force at pleasure, and so vary the velocity acquired.

Having stretched the cord at a small elevation, if we allow the weight to run down, and note the instant at which the motion commences; by means of a pendulum beating seconds, we can observe the spaces described in the first second, in the first two seconds, in the first three seconds, and so on. This being done, we shall find that the spaces described are proportional to the numbers 1, 4, 9, 16, 25, &c., i. e. to the squares of the times; which establishes the truth of equation (9), and proves that gravity is a constant force. The other laws of rectilinear motion may also be confirmed by the inclined plane of Galileo; but as a means of illustrating the principles of Dynamics, it is very inferior to the method devised by Mr. Atwood, the principle of which admits of an easy explanation, although the details of construction and precautions to be used in observing are unsuited to the limits of this work.

Atwood's Machine, when reduced to its most elementary form, consists of a pulley O (which in practice is



generally mounted on four friction wheels), and a cord passing over it, at the two extremities of which are suspended two unequal weights, P and Q; if these weights were equal there would be no motion, and the smaller the difference between them is, the less will be the motive force and consequent velocity: let f represent the Dynamical force which animates the system; the Statical force which produces this Dynamical force is evidently $P - Q$, i. e. the difference of the weights, and the mass moved is the sum of the masses contained in $P + Q$; if the weights P and Q were allowed to fall together by the force of gravity, they would acquire in one second a velocity equal to g, the Statical force producing this velocity being evidently $P + Q$, i. e. the sum of the weights; we have, therefore, by the second law of motion,

$$f : g :: P - Q : P + Q,$$

because the Dynamical forces are to each other as the Statical forces producing them; hence we obtain—

$$f = \frac{P - Q}{P + Q} g. \qquad (18)$$

It is evident that we can make the force f as small as we please by diminishing the difference $P - Q$; the motion produced by this force is therefore completely under our control, and may be made as slow as the convenience of observation requires. All the laws of rectilinear motion caused by constant forces, *vid.* equations (8), (9), (10), may be illustrated by means of this machine, with such accuracy as to leave no doubt of their truth on the learner's mind. Atwood's Machine and the Inclined Plane of Galileo are both based upon the principle of diminishing the force of gravity, so as to allow of direct observation of the motions produced. So far as *direct proof of the* laws of motion is concerned, no apparatus

can be better conceived than Atwood's Machine; but the student must remember that these experiments are here mentioned merely as illustrations, and not proofs, of the laws of motion. The most certain proofs of these laws, as we have already mentioned, are to be found in remote and indirect consequences, which admit of easy and accurate observation, and from which we can reason back rigorously to the laws of motion assumed at the commencement of the study of Dynamics.

Miscellaneous Exercises.

The equations (8), (9), (10) must be modified as follows, if the body to which the force is applied be already in motion with a velocity V,

$$v = V \pm ft, \qquad (8 \text{ bis})$$
$$s = Vt \pm \tfrac{1}{2}ft^2, \qquad (9 \text{ bis})$$
$$v^2 = V^2 \pm 2fs, \qquad (10 \text{ bis})$$

the positive or negative sign being used, according as the velocity acts with or against the force. By the aid of these equations we are enabled to solve many useful and interesting questions, examples of which will be given in the following exercises.

1. Find the height to which a body projected vertically with a velocity of 80 feet per second will ascend. When the body has reached its greatest height, its velocity v is nothing; therefore, making $v = 0$ in equation (10 bis), we find

$$0 = V^2 - 2fs = (80)^2 - 64s,$$

and therefore,

Ans. $s = 100$ ft.

2. A stone falls from the mouth of a coal-pit, and, 3 seconds after, another falls from a point 100 fath. lower down in the shaft. After how many seconds will the first stone overtake the second, and at what depth in the pit?

Let s denote the unknown depth traversed by the first stone at the time it overtakes the other, and let s' be the depth descended by the

second stone, and t the number of seconds measured from the starting of the second stone; then

$$s = 16\,(t + 3)^2$$
$$s' = 16\,t^2$$
$$s - s' = 600 \text{ ft.} = 16\,(6t + 9).$$

And finally

Ans. Depth $= 961$ feet.
Time $= 7\frac{3}{4}$ seconds.

3. If two weights P and P' rest on inclined planes, whose inclinations are i and i', and be joined by a string passing without friction over a pulley at the common vertex of the planes; find the acceleration acquired in one second by the weights.

Let f be the required acceleration; we then have the statical force, $P \sin i - P' \sin i'$, producing the acceleration f in the two bodies; but in the same time the weights $P + P'$ would produce the acceleration g; and, therefore, by the second law of motion,

$$\textit{Ans. } f = g\,\frac{P \sin i - P' \sin i'}{P + P'}.$$

4. A stone is projected vertically upwards with a velocity of 150 ft. per second, and, one second after, another stone is projected with a velocity of 200 ft. per second. *When* and *where* will the stones meet?

Equation (9 bis), for the first stone is

$$s = 150\,t - 16\,t^2,$$

and for the second

$$s = 200\,(t - 1) - 16\,(t - 1)^2.$$

Equating the values of s, we find, *Ans.* $t = 2.634$ secs.
$s = 284$ ft.

5. A horse exerting a traction T on a waggon whose weight is W, draws it up a hill, of height h and length l; find the time of ascending the hill?

$$\textit{Ans. } t = \sqrt{\frac{2}{g} \times \frac{W}{Tl - Wh}}.$$

6. A stone is projected vertically downwards with a velocity of 50 feet per second; if a second stone be projected after it in 10 seconds, when will it overtake the first stone, the velocity of projection of the second stone being double that of the first.

This question is solved by (9 bis).

The two equations are,

$$s = 50\,t + 16\,t^2,$$
$$s = 100\,(t - 10) + 16\,(t - 10)^2.$$

Equating the values of s, we find *Ans.* 2.22 seconds.

7. Two inclined planes whose lengths are l_1 and l_2, and common height h, are placed back to back, and two weights W_1 and W_2 rest upon them, connected by a string passing over a pulley placed at their common vertex; find the acceleration caused in either weight in one second. (*Newth.*)

$$\textit{Ans. } f = \frac{gh}{l_1\,l_2} \times \frac{W_1\,l_2 - W_2\,l_1}{W_1 + W_2}.$$

8. A stone, in falling, describes the n^{th} part of the height in the last second; find the time of falling.

$$\textit{Ans. } t = n \pm \sqrt{n\,(n-1)}.$$

9. A stone is thrown vertically down a cliff 300 ft. in height, and is observed to reach the base of the cliff in 4 seconds; what was the velocity of projection? *Ans.* 11 ft. per second.

10. A stone is thrown vertically upwards from the bottom of a tower 300 ft. high, with a velocity of 100 ft. per second; after what time should another stone be projected downwards from the summit of the tower, with the same velocity, in order that they may meet at the middle point of the tower? *Ans.* 1.25 seconds.

11. A stone is dropped from a height of 400 ft., and, when it has fallen through 50 ft., a second stone is projected after it, so that both stones reach the ground together; find the velocity of projection.

Ans. 72 ft. per second.

12. The time occupied by a body in running down an inclined plane is n times the time occupied by another body in falling down its height. Find the inclination of the plane. *Ans.* $\sin i = \frac{1}{n}$.

13. It is required to divide an inclined plane, whose length is l, into n parts, such that the times of describing the successive parts shall be equal. *Ans.* $\frac{l}{n^2}, \frac{3l}{n^2}, \frac{5l}{n^2}$, &c.

14. Find the velocity with which a body should be projected down an inclined plane, so that the time of running down the plane shall be equal the time of falling down the height.

$$\textit{Ans. } V = \frac{g\,(l - h \sin i)}{\sqrt{2gh}}.$$

15. Find the inclination of this plane, when a velocity of $\frac{3}{4}$ths that due to the height is sufficient to render the times of running down the plane, and of falling down the height, equal to each other. (*Wrigley.*)

Ans. 30°.

16. Through what chord of a circle, drawn from the bottom of the vertical diameter, must a body descend, so as to acquire a velocity equal to $\frac{1}{n}$th part of the velocity acquired in falling down the vertical diameter?

Ans. If θ denote the angle between the required chord and the vertical diameter, $\cos\theta = \frac{1}{n}$.

17. Find the radius of a circle, such that a body running down will describe the radius in the same time as another body requires to fall down the vertical diameter. (*Wrigley.*)

Ans. The radius required makes an angle of 60° with the vertical diameter.

18. Find the diameter down which a body falling will describe the last half in the same time as the vertical diameter. (*Wrigley.*)

$$\textit{Ans. } \cos\theta = \frac{3\sqrt{2} - 4}{2\sqrt{2}}.$$

$$\theta = 85^\circ\ 4'\ 43''.6.$$

19. A falling body describes one-third of the entire height in the last second; find the height, and time of descent.

Ans. Time = 5.449 seconds.
Height = 475 ft.

20. A body is projected vertically upwards with an unknown velocity, V; and having reached a known height, h, the time, t, is measured, which elapses until the body, in falling again, passes the point, whose height is h. Find the velocity of projection, and the total height to which the body ascends.

$$\textit{Ans. } V = \sqrt{2gh + \left(\frac{gt}{2}\right)^2}.$$

$$H = h + \frac{gt^2}{8}.$$

21. With what velocity must a body be projected downwards, so as in n seconds to overtake another body which has already fallen through a distance h? (*Wrigley.*)

$$\textit{Ans. } \text{Velocity} = \frac{h}{n} + \sqrt{2gh}.$$

22. A descending weight P draws a weight Q up an inclined plane, whose height and length are h and l, by a cord passing over a pulley at the summit of the plane; find when the cord should be cut, in order that the weight Q may just ascend to the top of the plane. (*Wrigley.*)

Ans. The weight P must have reached a distance from the bottom of the plane $= \frac{P+Q}{P} \times \frac{hl}{h+l}$.

23. A body is projected vertically downwards with a velocity V_1; when must a second body be projected with a velocity V_2, so as to overtake the first body in t seconds after the projection of the former? (*Newth.*)

$$Ans.\ \frac{1}{g}\left\{(gt + V_2) \pm \sqrt{g^2t^2 + 2gV_1t + V_2^2}\right\}.$$

24. A weight of 30 lbs., resting on an inclined plane of 60°, draws (by means of a string passing over the common vertex of the two planes), another weight of 20 lbs., resting on a plane of 40° inclination. Find the velocity acquired in the first second of the motion, and the tension of the string. *Ans.* $v = 8.39$ ft. per second, $T = 18.108$ lbs.

25. If two weights, P, P', resting upon planes i, i', and joined as in the last example, be in motion; find the tension of the string.

$$Ans.\ \frac{PP'}{P+P'}(\sin i + \sin i').$$

26. A weight Q, resting on a horizontal table, is drawn along the table by a weight P, attached to Q by means of a horizontal cord passing over a pulley placed at the edge of the table. If μ denote the coefficient of friction, find the tension of the cord.

$$Ans.\ \frac{PQ}{P+Q}(1+\mu).$$

27. If in example (25) both weights rub with a friction, whose coefficient is μ against the planes; find the tension of the string.

$$Ans.\ \frac{PP'}{P+P'}\left((\sin i + \sin i') + \mu(\cos i' - \cos i)\right).$$

28. A body is projected along a horizontal surface with a velocity of 20 feet per second, and is brought to rest in 400 yards. Find the coefficient of friction, and the duration of the motion.

$$Ans.\ \mu = \frac{1}{192},$$
$$t = 2 \text{ minutes}.$$

29. Find the inclination of Galileo's plane, which will give a velocity of one foot per second, in one second, if the friction be $\mu = \frac{1}{3}$.

Ans. 20° 8'.

CHAPTER IV.

ON UNIFORM CIRCULAR MOTION.

1. Centrifugal Force.—2. Diurnal Rotation of the Earth.

1. **Centrifugal Force.**—By the first law of motion it appears that a body in motion, not acted on by any force, will continue to move uniformly in a right line; consequently, any variation in its velocity, or any curvature in its path, is a sure sign that some external force acts upon it.

In the last chapter we considered the alteration of velocity produced by a constant force acting in the direction in which the motion takes place. In the present chapter we shall consider the mode of estimating the amount of the force which causes the path of a body to deviate from a right line, in the case of uniform circular motion.

Let VOO′C be a circle described by a body in uniform motion, and let a regular polygon VOO′, &c., be inscribed in the circle, whose sides are so small that we may assume the polygon to coincide with the circle; and let the unit of time be so chosen that each side VO, OO′ is described by the revolving body in the unit of time; VO or OO′, therefore, represents the velocity of the body. The body, having travelled from V to O in the unit of time, would in the next unit describe the line OA equal to VO, but at the point O it is impelled by a force in the direction OC, which compels it to describe the line OO′ instead of OA; therefore, completing the parallelogram OAO′B, the line OB represents the deflecting force, and OX is half the deflecting force; but in the right-angled triangle COO′, OO′ is a mean proportional between CO *and OX; therefore*, if the deflecting force, the velocity,

and radius of circle be called f, v, r, respectively, we have :—

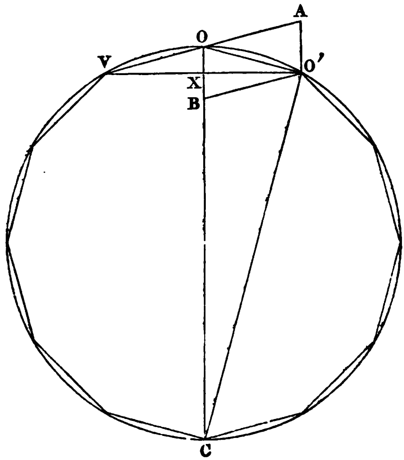
Since,

$$(OO')^2 = OC \times OX,$$

$$v^2 = 2r \times \frac{f}{2},$$

and finally

$$f = \frac{v^2}{r}. \qquad (19)$$



The centrifugal force, which is equal and opposite to the deflecting force, is therefore directly proportional to the square of the velocity, and inversely proportional to the radius of the circle described.

The equation (19) may be put into a more convenient form, in the following manner. Let T denote the time of describing a complete revolution, called the Periodic Time, and let π represent the ratio of the circumference

of a circle to its diameter, i. e. 3.14159; the circumference of the circle described will be $2\pi r$, which is the space described in the time T. By equation (1) the velocity is equal to the space divided by the time, i. e.

$$v = \frac{2\pi r}{T};$$

hence, substituting this value in equation (19), we find

$$f = 4\pi^2 \frac{r}{T^2}; \tag{20}$$

i. e. the *centrifugal force is directly proportional to the radius of circle described, and inversely proportional to the square of the Periodic Time.*

In equations (19), (20), the force f is understood to be the force applied to the unit of mass. If it were necessary to compare the effects of centrifugal force upon different bodies, we should then multiply each centrifugal force by the mass of the body it is applied to move.

EXAMPLES.

1. If a body be compelled to move on a circle with a velocity of 300 yards per minute, the radius of the circle being 16 feet, calculate the centrifugal force. *Ans.* 14.062 ft. per sec.

2. If a body be compelled to move on a circle, whose radius is 17 yards, with a velocity of 361 feet per second, calculate the centrifugal force. *Ans.* 2555.3 ft. per sec.

3. If a body, weighing 17 tons, move on the circumference of a circle, whose radius is 1110 feet, with a velocity of 16 feet per second, calculate the centrifugal force in tons.

The centrifugal force in tons : weight of body in tons : : the centrifugal force in feet $\left(=\frac{v^2}{r}\right)$: gravity in feet; i. e.

Centrifugal force in tons : 17 tons : : $\frac{256}{1110}$: 32.1948.

Hence, centrifugal force equal to . . . . 0.1217 ton.

4. If a body, weighing 1000 lbs., be constrained to move in a circle, whose radius is 100 feet, by means of a string, capable of sustaining a strain not exceeding 450 lbs., calculate the velocity at the moment the string breaks. *Ans.* 38.06 ft. per sec.

5. If a railway carriage, weighing 7.21 tons, moving at the rate of 30 miles per hour, describe a portion of a circle, whose radius is 460 yards, calculate its centrifugal force in tons.

Ans. 0.314 ton.

6. Two bodies, weighing respectively 13 lbs. and 16 lbs., move with different velocities on circles whose radii are 11 feet and 15 feet respectively: find the ratio of their velocities when the centrifugal forces exerted upon them are equal. *Ans.* $\frac{29}{21}$ nearly.

7. The time occupied by a body in describing uniformly a complete revolution in a circle, whose radius is 11 feet, is 16 seconds, calculate the centrifugal force which acts upon it. *Ans.* 1.696 ft. per sec.

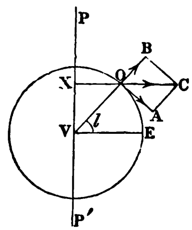
8. If the time occupied in describing 61° of the same circle be 11 minutes, calculate the centrifugal force. *Ans.* 0.000028 ft. per sec.

9. If the radius of the circle be 11 yards, and the periodic time 4 minutes, calculate the centrifugal force. *Ans.* 0.0226 ft. per sec.

10. If the centrifugal force, in a circle of 100 feet radius, be 146 feet per sec., find the periodic time. *Ans.* 5.2 secs.

11. If the centrifugal force be 131 ounces, and the radius of the circle 100 feet, the periodic time being one hour, calculate the weight of the body. *Ans.* 386.309 tons.

2. **Diurnal Rotation of the Earth.**—In the diurnal rotation of the earth on its axis we have a remarkable example of centrifugal force; in this case the periodic time is the same for all parts of its mass, and consequently every particle of the matter composing the earth is repelled from the axis of rotation by a force proportional to its distance from that axis (*vid.* equation 20). It is plain that the centrifugal force arising from the earth's rotation diminishes



the attraction which is exerted by the earth on all bodies

at its surface. We shall first calculate the effect produced by centrifugal force at the equator, and then describe its effects at other points on the earth's surface. Let PP′ be the axis on which the earth revolves, and VE the equatorial radius equal to R; let G denote the attraction of the earth uninfluenced by rotation, and g as usual denote the *actual* gravity; if f represent the centrifugal force at the equator, we shall have

$$f = G - g;$$

f is known, from equation (20), by substituting for r, the earth's equatorial radius, 20923596 feet, and for T, the number of seconds in a sidereal day, 86164^s; we thus obtain $f = 0.11126$ ft. per sec., and since g (the force of gravity at the equator) is about 32.088 ft. per sec., we find,

$$f = \frac{g}{288.40};$$

and finally,

$$f = \frac{G}{289.40}. \qquad (21)$$

This may be expressed in whole numbers, by stating that the force of the earth's attraction is 289 times the centrifugal force at the equator.

The effect of the earth's rotation at the other points of its surface may be thus found:—Let O be any point on the surface, OE being its latitude, represented by l; draw OX perpendicular to the axis of rotation PP′; make OC equal to the required centrifugal force; and call this unknown force ϕ; and then resolve the force OC into two components OB and OA, one in the prolongation of the earth's radius, and the other perpendicular to it; the component OB is employed in diminishing gravity, and its amount may be thus calculated: OB is equal to *OC multiplied* by the cosine of BOC, which is equal to

the latitude BVE : i. e. $OB = \phi \cos l$; but by equation (20), the centrifugal force ϕ is

$$\phi = \frac{4\pi^2}{T^2} \times OX;$$

therefore

$$OB = \frac{4\pi^2}{T^2} \times OX \cos l;$$

but if R denote the radius VO, we have $OX = R \cos l$, and therefore finally OB, or

$$\text{The component of centrifugal force diminishing gravity} = \frac{4\pi^2 R}{T^2} \cos^2 l. \qquad (22)$$

The part of centrifugal force which is employed in diminishing gravity is thus proved to *vary as the square of the cosine of the latitude.* If the latitude be equal to zero, i. e. if the point be situated on the equator, $\cos l$ will be equal to unity, and the centrifugal force will be equal to

$$\frac{4\pi^2 R}{T^2};$$

which has been already shown to be equal to 0.11126 feet per sec.; introducing this coefficient, we obtain for the surface of the earth the equation

$$OB = 0.11126 \cos^2 l.$$

The component of centrifugal force OA, perpendicular to the radius of the earth, is employed in transporting the materials of which the earth's crust is composed towards the equator; this is evident from the direction in which it acts; in consequence of this transporting force, the shape of the earth, even though it were supposed to have been originally spherical, would tend to become somewhat flattened at the poles, and protruding at the

equator; the amount of the transporting force is easily calculated: OA is equal to OC multiplied by the sine of BOC or latitude.

Therefore

$$OA = \phi \sin l,$$

or, since

$$\phi = \frac{4\pi^2}{T^2} \times OX, \text{ and } OX = R \cos l,$$

we find OA, or

$$\left.\begin{array}{l}\text{Force transporting materials towards}\\ \quad\text{the equator}\end{array}\right\} = \frac{4\pi^2 R}{T^2} \sin l \cos l. \qquad (23)$$

or, for the surface of the earth,

$$OA = 0.11126 \sin l \cos l.$$

It is easy to prove from this equation that the transporting force will be equal to zero at the equator and at the poles, and will have its greatest value at the latitude of 45°, when it becomes equal to half the centrifugal force at the equator.

Exercises on Centrifugal Force.

1. Calculate the effect of centrifugal force in diminishing gravity in the latitude 60°. By equation (22) the component of centrifugal force employed in diminishing gravity is equal to $\cos^2 60°$ multiplied by 0.11126; $\cos^2 60° = 0.25000$; hence we obtain the component of centrifugal force, = 0.0278 ft. per sec.

2. Calculate, for the same latitude, the component of the centrifugal force employed in transporting the materials of the earth's crust to the equator. This is done by means of equation (23). Sin 60° = 0.86602; cos 60° = 0.50000; multiplying the product of these quantities by 0.11126, we obtain finally the transporting force equal *to* 0.04817 ft. per sec.

3. Calculate the diminution of gravity caused by centrifugal force at the latitude of 23°. *Ans.* 0.09 ft. per sec.

4. Calculate the transporting force at the same latitude. *Ans.* 0.04 ft. per sec.

5. Calculate both these forces for the latitude of Dublin, viz., 53° 21′.

Force diminishing gravity = 0.0396441 ft. per sec.
Transporting force = 0.0532837 ft. per sec.

6. Calculate the diminution of gravity at the Sun's equator caused by centrifugal force, the radius of the Sun being 441000 miles, and the time of revolution on his axis being 607 h. 48 m. *Ans.* 0.019200 ft. per sec.

7. Calculate the centrifugal force at the equator of Mercury, the radius being 1570 miles, and the time of revolution 24 h. 5 m. *Ans.* 0.04353 ft. per sec.

8. Find centrifugal force at the equator of Venus; radius being 3900 miles, and time of revolution 23 h. 21 min. *Ans.* 0·11504 ft. per sec.

9. The radius of the planet Mars is 2050 miles, and his periodic time 24 h. 37 m., find centrifugal force at the equator. *Ans.* 0.05441 ft. per sec.

10. The radius of Jupiter is 43500 miles, and periodic time is 9 h. 56 m.; find centrifugal force at equator. *Ans.* 7.09068 ft. per sec.

11. The radius of Saturn is 39580 miles, and periodic time is 10 h. 29 m.; find centrifugal force at equator. *Ans.* 5.79243 ft. per sec.

12. The radius of Uranus is 17250 m., and the periodic time 9 h. 30 m.; find the centrifugal force at his equator. *Ans.* 3.0741 ft. per sec.

13. The Moon's distance from the Earth is 59.96435 radii of the Earth; the Earth's diameter is 7925.6 m., and the periodic time of the Moon is 27 d. 7 h. 43 m. 11 s.; find the centrifugal force of the Moon in her orbit. *Ans.* 0.0088891 ft. per sec.

14. Supposing the Earth's attraction to diminish as the square of the distance increases; find what it becomes at the distance of the Moon, gravity being 32.2 ft. *Ans.* 0.0089551 ft. per sec.

15. Find a superior limit to the height of the Earth's atmosphere at the equator, by making the centrifugal force equal to gravity. *Ans.* 22250 miles.

16. If a body move uniformly, with a velocity v, round the sides of a regular polygon of n sides, by virtue of a blow given at each corner in the direction of the bisector of the angle; find its amount.

Ans. It must be such as would generate in the moving body a velocity,

$$f = 2v \sin \frac{\pi}{n}.$$

17. An equilateral triangle, whose sides are each 50 ft., is described by a body of 14 lbs., in 10 seconds; find the amount of the blow that must be given at each corner.

Ans. Force of blow = 11.36 lbs.

18. If this blow be given by a hammer moving at the rate of 1 ft. per second, what must be its weight? *Ans.* 365.79 lbs.

19. If a pendulum, whose length is l, be made to move on a horizontal circle, so as to make an angle θ with the vertical; find its velocity and time of revolution?

N. B.—This pendulum is sometimes called conical, and occurs in the governor of the steam engine.

$$\textit{Ans.} \quad v = \sqrt{gl \sin \theta \tan \theta}.$$

$$T = 2\pi \sqrt{\frac{l \cos \theta}{g}}.$$

20. A heavy body, suspended by a flexible cord, revolves in a vertical circle; find under what circumstances may the strain upon the cord vanish.

Ans. It is easy to show that there exists a certain horizontal line from which, if the body were let fall, it would have the same velocity that it has in the point of the circle corresponding to its intersection; let this line be drawn. Draw a perpendicular from it to intersect the given circle, such that the intercept between the line and circle shall be one-fourth part of the chord of the circle. At this point of the circle the strain on the cord vanishes.

21. Balls, weighing each 10 lbs., are fixed at the ends of a rod 8 ft. long, revolving 100 times per minute round a central vertical axis. Find the tension of the connecting rod.

Ans. 137.08 lbs.

22. What should be the time of rotation of the earth, in order that bodies at the equator should have no weight?

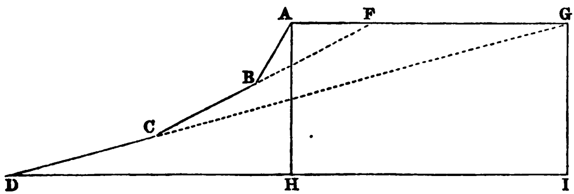
Ans. The day should be one-seventeenth part of its present length.

CHAPTER V.

ON THE PENDULUM.

1. Motion of a Body down a System of Inclined Planes.—2. Velocity acquired by a Heavy Body in falling down a Circular Arc.—3. Time of falling down a Circular Arc.—4. The Simple Pendulum.—5. Acceleration due to Change of Place.—6. Acceleration due to Change of Length.

1. **Motion of a Body down a System of Inclined Planes.**—If a heavy body fall down a system of inclined planes, and if we suppose that it loses no velocity in passing from one plane to another, it may be proved that the final velocity is equal to that which would be acquired in falling down the height.

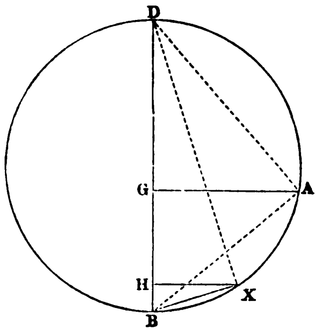


Let the system of planes be represented by the lines AB, BC, and CD, and its height by AH. By equation (16), the velocity acquired in falling down AB is equal to that which would be acquired in falling down the plane FB, since they have the same height. And if we suppose the body to enter the plane BC, without any loss of velocity in passing from AB to BC, it is evident that the velocity acquired at the point C will be equal to that which would be acquired in falling down GC. And finally, if it enter the plane CD without loss of velocity, it will arrive at D with the same

velocity as if it had descended from G along the plane GD; and as the height of this plane is equal to that of the system, it follows from (16), that if v be the final velocity,

$$v^2 = 2g \times \text{height}.$$

2. **Velocity acquired by a Heavy Body in falling down a Circular Arc.**—Let AX be an arc of a circle whose plane is vertical, and let AG and XH be horizontal lines drawn to meet the vertical diameter in the points G and H; if a heavy body descend from A to X along the arc AX, its velocity at X will be equal to that which would be acquired in falling down the height GH. For if AX be divided into an indefinitely great number of small arcs, the whole may be considered as a system of inclined planes along which the heavy body will pass successively without any loss of velocity. The velocity at X, therefore, will be that due to the height, and therefore



$$v^2 = 2g \times GH.$$

If the radius of the circle be denoted by l, and the chords AB and XB by a and x respectively, it follows, because the triangles BAD and BXD are right-angled, that $BG = a^2 \div 2l$, and $BH = x^2 \div 2l$, and, therefore, that $GH = BG - BH = (a^2 - x^2) \div 2l$. Substituting this value of GH, we obtain

$$v^2 = \frac{g}{l}(a^2 - x^2). \tag{24}$$

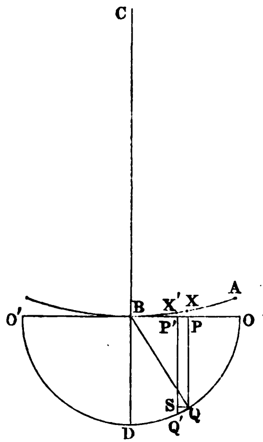
EXAMPLES.

1. If the radius of the circle be 72 feet, and the body fall from a distance of 50 feet from the lowest point, until it reach a distance of 30 feet; calculate the acquired velocity, if $g = 32$.

Ans. $26\frac{2}{3}$ ft. per sec.

2. If the radius be 100 feet, and if the body fall from the extremity of the horizontal diameter; calculate the velocity, after describing half the quadrant, if $g = 32.1908$.   *Ans.* 67.47 ft. per sec.

3. **Time of falling down a Circular Arc.**—The problem of expressing the time of a heavy body falling down a circular arc is incapable of solution in finite terms, except in the following case, which alone is of practical importance.



Let the point A, from which the descent commences, be so near the lowest point B, that the length a of the arc BA does not differ sensibly from its chord, and *a fortiori* the length x of the arc BX from its chord. With B as centre, and the length BA as radius, describe a semicircle cutting the tangent drawn at B, in the points O and O′; on BO assume BP = BX, X being the point at which the heavy body has arrived. After an indefinitely short period, let it arrive at X′, and let P′B be assumed = BX′, draw the lines PQ and P′Q′ perpendicular, and QS parallel to BO, and join BQ; then, since $a^2 - x^2 = BQ^2 - BP^2 = PQ^2$, we may express the velocity of the heavy body at X, by substi-

tuting in equation (24) for $a^2 - x^2$ its value PQ^2, we thus obtain

$$v = \sqrt{\frac{g}{l}} \times PQ.$$

Through the indefinitely small arc XX′ we may suppose the velocity constant, and therefore, by equation (1), the time of describing it may be expressed by space $\div$ velocity = XX′ $\div$ v, and therefore

$$\text{Time of describing } XX' = \sqrt{\frac{l}{g}} \times \frac{XX'}{PQ}.$$

From the similar right-angled triangles BQP and Q′QS, we have QS : QQ′ : : PQ : BQ. Or, substituting for QS and BQ their equals XX′ and a respectively, XX′ : QQ′ : : PQ : a; and therefore *alternando* (Euc. v. 16) XX′ : PQ : : QQ′ : a; and XX′ $\div$ PQ = QQ′ $\div$ a. Substituting this in the last equation, we find

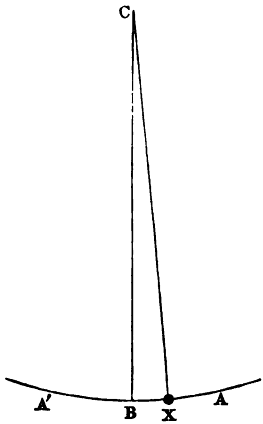
$$\text{Time of describing } XX' = \sqrt{\frac{l}{g}} \times \frac{QQ'}{a}.$$

If the whole arc be divided into an indefinitely great number of elements similar to XX′, the time of falling from A to B will be equal to the sum of the times of describing each element separately. But the sum of the elements QQ′ of the quadrant OD $\div$ its radius, is equal to the length of the quadrant $\div$ $a = \frac{1}{2}\pi$ (Manual of Trigonometry). And therefore,

$$\text{Time of describing } AB = \frac{\pi}{2}\sqrt{\frac{l}{g}}. \qquad (25)$$

4. **The Simple Pendulum.**—A heavy body X suspended from a fixed point C by a fine string or rod, which vibrates in a *small* circular arc on each side of the lowest point B, is called a *simple pendulum.* When the heavy body, which is usually called the *bob* of the *pendulum,* descending from A, has reached the lowest

point B, with a velocity due to height of A above B, it will commence to ascend the opposite arc with this velocity, and will continue to ascend with a gradually diminishing velocity, until it reaches A′ at an equal height above B, from which it will again descend towards B, and would thus continue swinging to and fro through equal arcs for ever, if it were not checked, and gradually brought to rest, by such forces as friction, rigidity in the mode of suspension, resistance of the air, and such like. It is evident that T, the time of one oscillation, is double the time of descending from A to B, and therefore, by equation (25) we have



$$T = \pi\sqrt{\frac{l}{g}}. \qquad (26)$$

As the value of T in this equation depends solely on the length of the pendulum and the force of gravity, it appears, that for small arcs *the duration of an oscillation is independent of the length of the arc of vibration:* from this property the vibrations of a pendulum are said to be *isochronous.* Solving the last equation for l, we obtain

$$l = \frac{gT^2}{\pi^2}.$$

As g in this equation signifies twice the number of feet which a heavy body falls through in a second, l must express the number of feet in a pendulum, the duration

of whose vibration is T seconds. If L denote the length of a *seconds pendulum*, i. e. of a pendulum which oscillates once in every second, we obtain its length by making $T = 1$ in the last equation, which gives

$$L = \frac{g}{\pi^2}. \qquad (27)$$

Substituting this value of $g \div \pi^2$ in the expression for l, we obtain

$$l = LT^2. \qquad (28)$$

From which it follows, that *the length of a pendulum is proportional to the square of the number of seconds in one vibration.*

From the Table in page 104, it appears that in London $g = 32.1908$ ft. per sec.; substituting this number in equation (27), and reducing the result to inches, we obtain

Length of a seconds pendulum in London = 39.14 inches.

From equation (27) we obtain the value of the dynamical measure of gravity in terms of the length of the seconds pendulum,

$$g = \pi^2 L. \qquad (29)$$

This furnishes the most accurate method of determining the values of gravity at different parts of the Earth's surface. The Table in page 104 has been constructed from observations made on the length of the seconds pendulum.

EXAMPLES.

1. From the value of g in Table, page 104, calculate the length of the seconds pendulum in Paris. *Ans.* 39.128 inches.

2. From the experiments of Kater and Sabine, the length of the seconds pendulum in London has been shown to be 39.139 inches; from this calculate the dynamical measure of gravity.
Ans. 32.1905 feet per sec.

3. Calculate the length of a seconds pendulum at Spitzbergen.
Ans. 39.2144 inches.

4. Calculate the length of the seconds pendulum at the Isle of Rawak, on the Equator. *Ans.* 39.0144 inches.

5. Dent's clock, which is erected in the clock-tower of the Houses of Parliament, Westminster, has a pendulum which vibrates once in two seconds; calculate its length. *Ans.* 13.046 feet.

6. Calculate the length of a half-seconds pendulum in London.
Ans. 9.78 inches.

From the experiments on the length of the seconds pendulum at various latitudes, north and south of the Equator, made for the English Government by Captain Kater and Colonel Sabine, and for the French Government by MM. Borda, Biot, Mathieu, Bouvard, Arago, Chaix, and Captains Duperrey and Freycinet, the following simple relation may be deduced between the length L of a seconds pendulum at any latitude λ, and 39.118, the length of a seconds pendulum at the mean latitude 45°,

$$L = 39.118 - \tfrac{1}{10} \cos 2\lambda. \qquad (30)$$

EXAMPLES.

1. Calculate the length of the seconds pendulum, and the dynamical measure of gravity, in Dublin, at the latitude of 53° 20′ N.
Ans. $L = 39.146$ in.
$g = 32.196$ ft. per sec.

2. Calculate the length of the seconds pendulum, and the dynamical measure of gravity, in Edinburgh, at the latitude of 55° 57 N.
Ans. $L = 39.155$ in.
$g = 32.204$ ft. per sec.

3. Calculate the length of the seconds pendulum, and the dynamical measure of gravity, at the Equator, and at the Pole.
Ans. At the Equator, $L = 39.018$. in.
$g = 32.091$. ft. per sec.
At the Pole, $L = 39.218$. in.
$g = 32.255$ ft. per sec.

5. **Acceleration due to Change of Place.**—If n be the number of vibrations made by a pendulum in a mean solar day, which contains 86400 seconds, then $T = 86400 \div n$. Substituting this value in equation (28) we obtain

$$n^2 = \frac{L}{l} \times (86400)^2. \tag{31}$$

If we transport the pendulum l to another latitude, where the length of the seconds pendulum is L', and the number of vibrations n', we have

$$n'^2 = \frac{L'}{l} \times (86400)^2;$$

and by division,

$$\frac{n'^2}{n^2} = \frac{L'}{L},$$

subtracting each side from unity, we obtain

$$\frac{n^2 - n'^2}{n^2} = \frac{L - L'}{L};$$

but $n^2 - n'^2 = (n + n')(n - n') = 2n(n - n')$ as n and n' are very nearly equal; making this substitution, and solving, we obtain for $(n - n')$ the acceleration of the number of vibrations of a pendulum for one solar day, consequent on its removal from one latitude to another, the following value,

$$n - n' = \frac{n}{2} \times \frac{L - L'}{L}. \tag{32}$$

and as L by equation (29) is proportional to g,

$$n - n' = \frac{n}{2} \times \frac{g - g'}{g}. \tag{33}$$

EXAMPLES.

1. *Calculate the retardation in a pendulum beating seconds at London, if it be transported to the Equator.* *Ans.* 133.5 seconds.

2. Colonel Sabine observed that the number of vibrations of a pendulum in London, in a mean solar day, was 85945.80. Having conveyed the same pendulum to Paris, he observed the number of vibrations to be 85933.83. From this calculate the difference of gravity, in London and Paris, as a fraction of its total amount.

Ans. $\frac{1}{3590}$.

3. From the experiments made by Mr. Airy, the Astronomer-Royal, in the Harton coal-pit, 1260 feet deep, near South Shields, it was ascertained that a pendulum beating seconds at the surface, if conveyed to the bottom of the pit, gained $2\frac{1}{4}$ seconds in a day. From this calculate the increase of gravity at the bottom of the pit, as a fraction of its total amount. *Ans.* $\frac{1}{19200}$.

6. **Acceleration due to Change of Length.**—If T be the time of vibration of a pendulum whose length is l, it follows from equation (28) that

$$T^2 = \frac{l}{L}.$$

If l become l', and the corresponding vibration T' seconds, a number differing very little from T, if the increase of length be inconsiderable,

$$T'^2 = \frac{l'}{L},$$

by division

$$\frac{T'^2}{T^2} = \frac{l'}{l}.$$

Subtracting unity from each side, and making reductions similar to those used in deducing equation (28), we obtain for the increase of time of vibration due to an increase of length the following expression,

$$T' - T = \frac{T}{2} \times \frac{l' - l}{l}. \qquad (34)$$

The adjustment of a pendulum to its proper length is generally effected by a nut on the end of the rod, by which the bob can be screwed up or down. By lengthening the pendulum we increase the duration of a vibra-

tion, or diminish the number of swings in a mean solar day, and thus retard its rate; by shortening, we accelerate its rate. In equation (34) $T' - T$ is the error of time in one vibration due to a change of length; if this error accumulate for a day which contains 86400 seconds, we obtain

$$\text{Error in a day} = 43200 \times \frac{l' - l}{l}. \tag{35}$$

We can ascertain, either by calculation or by trial, the effect of one turn of the nut; fractional parts of one turn are generally estimated by dividing the rim of the nut, and placing an index over it, so that we can elevate or depress the bob through a distance equal to one thread, one-half, one-quarter, &c., of a thread, at pleasure.

It is a well-known fact that bodies are increased in length by heat, and shortened by cold; from this it follows, that the rate of a clock regulated by a pendulum is retarded in hot weather, and accelerated in cold weather. We can correct this error by shortening or lengthening the pendulum by a suitable amount, which may be calculated if we know the increase of length due to a given change of temperature. We subjoin a Table of increase of length of materials which have been used in the construction of pendulums, due to a change of 10° Fahrenheit:

Value of $\frac{(l' - l)}{l}$ *for* 10° F.	
White deal,	$1 \div 41000$
Flint glass,	$1 \div 21000$
Steel rod,	$1 \div 15600$
Iron rod,	$1 \div 14400$
Brass,	$1 \div 9600$
Lead,	$1 \div 6300$
Zinc,	$1 \div 6120$

EXAMPLES.

1. If a pendulum swinging in London be 45 inches long, by how much will its rate be accelerated in one day if the bob be screwed up one turn, the screw having 32 threads to the inch?

Ans. 30 seconds.

2. If a clock lose two minutes a day, how many turns to the right hand must we give the nut in order to correct its error, supposing the screw to have 50 threads to the inch? *Ans.* 5.4 turns.

3. A mean solar day contains 24 hours, 3 minutes, 56.5 seconds, sidereal time; calculate the length of the pendulum of a clock beating sidereal seconds in London.

Ans. 38.925 inches.

4. Calculate the retardation in one day produced by an increase of 1° F. of temperature, on a brass seconds pendulum.

Ans. 0.45 seconds.

5. Calculate the acceleration in one day produced by a decrease of 10° F. of temperature, in an iron seconds pendulum.

Ans. 3 seconds.

6. In a seconds pendulum made of white deal, calculate the retardation in a day due to an increase of 13° F of temperature.

Ans. 1.37 seconds.

7. Calculate the effect produced on the rate per diem of a zinc pendulum, by a fall of temperature of 14° F. *Ans.* = 9.88 seconds.

8. The mean winter temperature of Dublin is 42° F., the mean summer temperature, 59.5° F. If a clock, regulated by a seconds pendulum of steel rod, be adjusted in winter, calculate its weekly loss in summer. *Ans.* 33.9 sec.

9. A heavy ball, suspended by a fine wire, vibrates in a small arc; 48 vibrations are counted in 3 minutes. Calculate the length of the wire. *Ans.* 45.87 feet.

10. The height of the cupola of St. Paul's, above the floor, is 340 feet; calculate the number of vibrations a heavy body would make in half an hour, if suspended from the dome by a fine wire which reaches to within six inches of the floor. *Ans.* 176.4.

CHAPTER VI.

ON THE COLLISION OF BODIES.

1. Elasticity of Bodies.—2. Impact of Bodies upon Plane Obstacles. —3. Collision of Bodies in Motion.

1. **Elasticity of Bodies.**—In the present chapter we propose to discuss various cases of the interesting questions which arise from the shock or collision of bodies. All bodies, whatever be their character in other respects, are subject to Newton's Third Law of Motion, which expresses that action and reaction are equal, or, in other words, when two bodies come into collision, that the momentum lost by one is gained by the other. In solving the problem of the collision of bodies, it is, however, necessary to take account of another property which has not been hitherto mentioned, viz., their elasticity. With respect to elasticity, bodies are divided into three classes:—

1. Perfectly elastic.
2. Imperfectly elastic.
3. Perfectly inelastic.

1. A body *perfectly elastic* is one that, if it impinges perpendicularly on a fixed plane, will recoil back from the plane with equal velocity, or the *velocity of recoil equals the velocity of approach.*

2. A body *imperfectly elastic* is one whose *velocity of recoil is less than its velocity of approach in a constant ratio.* And this constant ratio is called the *coefficient of elasticity.*

3. A *perfectly inelastic* body is one that does not recoil at all.

We may express these three kinds of elasticity in one statement, by saying that there is a coefficient of elasticity proper to each body in nature; that this coefficient cannot be less than zero, nor greater than unity; that when it is zero, the body is perfectly inelastic; when it is unity, the body is perfectly elastic; and when it has any intermediate value, the body is imperfectly elastic.

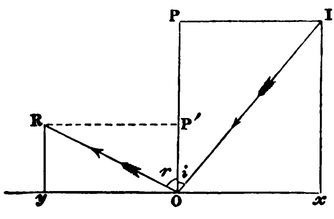
We shall throughout this chapter consider the case of imperfectly elastic bodies as the usual case, and deduce as particular examples the cases of perfectly elastic and inelastic bodies.

2. Impact of Bodies upon Plane Obstacles.—When a body impinges perpendicularly upon a plane, if the velocity of approach be called V, the velocity of recoil v, and the coefficient of elasticity e, we have the following equation,

$$v = eV, \tag{36}$$

and it has been ascertained by numerous experiments, that the ratio of v to V, or coefficient of elasticity, is constant, and independent of the magnitude of the velocity of approach.

If the body impinge upon the plane obliquely, its motion after reflexion may be thus found. Let IO be the path described by the impinging body in one second, and OR the path described by the body after reflexion in one second. Resolve the velocity IO or V into its two components,



$$xO = V \sin i,$$
$$PO = V \cos i.$$

In like manner resolve the reflected velocity v, or OR, into its components,

$$Oy = v \sin r,$$
$$OP' = v \cos r.$$

As the velocity parallel to the plane is not affected by the shock, we must have xO and Oy equal, and since OP and OP′ are the velocities of approach and recoil, they must have a constant ratio to each other; we thus find

$$v \sin r = V \sin i,$$
$$v \cos r = e \times V \cos i.$$

Dividing one of these equations by the other, we find

$$\tan i = e \tan r, \qquad (37)$$

from which, being given the angle of incidence i, and the coefficient of elasticity e, we can readily calculate the angle of reflexion r.

In order to find the velocity of the body after reflexion, we must square both equations, and add them together, from which results,

$$v^2 = V^2 (\sin^2 i + e^2 \cos^2 i). \qquad (38)$$

If we make $e = 1$, in equations (37) and (38), we shall have

$$\tan r = \tan i,$$
$$v^2 = V^2,$$

showing that in the case of a perfectly elastic body, the angle of incidence is equal to the angle of reflexion, and that the velocity is unaltered by the shock.

If, on the other hand, we make $e = 0$, we find

$$r = 90^\circ,$$
$$v = V \sin i,$$

equations which prove that a perfectly inelastic body

impinging obliquely on a plane, will, after impact, run along the plane, with a velocity diminished in the ratio of unity to sin i.

EXAMPLES.

1. If an imperfectly elastic ball impinge on a plane at an angle of 30°, and be reflected at an angle of 60°, find the coefficient of elasticity. *Ans.* $e = \frac{1}{3}$.

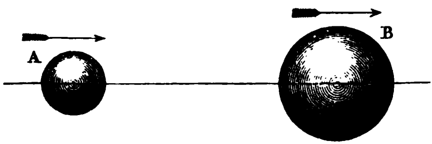
2. A ball moving at the rate of 10 miles per hour impinges at an angle of 20° upon a plane, the common coefficient of elasticity being 0.24; find its direction and velocity after reflexion.

Ans. $r = 56° \, 36'$.
$v = 4.096$ miles per hour.

3. A billiard-ball is driven from the side of a billiard-table, strikes the opposite side, and returns to the first side; prove that the times of going and returning are in the proportion of e to unity. (*Wrigley.*)

3. **Collision of Bodies in Motion.**—There are two cases of collision of bodies in motion to be considered: first, where the bodies before and after impact move in the same line; and secondly, where the bodies before and after impact move in different lines. The first case is called the Direct, and the second the Oblique, collision of bodies.

Direct Collision.—Let A and B be two bodies moving in the same line and in the same direction; let M, V, v,



denote the mass, velocity before, and velocity after, impact of A; and M', v', V', denote the corresponding quantities belonging to B.

The momentum lost by A in the shock is $M\,(V - v)$,

the momentum gained by B is $M'(v' - V')$; therefore, by the Third Law of Motion,

$$M(V - v) = M'(v' - V'); \qquad (39)$$

also, since $V - V'$ is the velocity of approach, and $v' - v$ the velocity of recoil; we have by the Law of Elasticity,

$$v' - v = e(V - V'). \qquad (40)$$

In equations (39) and (40) there are two unknown quantities v and v'; solving for these, we find

$$\left.\begin{aligned} v &= \frac{MV + M'V' - eM'(V - V')}{M + M'}; \\ v' &= \frac{MV + M'V' - eM(V' - V)}{M + M'}. \end{aligned}\right\} \qquad (41)$$

If the bodies be perfectly elastic, we must make $e = 1$ in (41); and if we suppose the bodies equal in weight, we find,

$$\begin{aligned} v &= V', \\ v' &= V, \end{aligned} \qquad (42)$$

from which it follows that if two perfectly elastic and equal balls come into collision, they *exchange velocities* during the shock.

If the bodies be perfectly inelastic, $e = 0$, and (41) become,

$$v = v' = \frac{MV + M'V'}{M + M'}, \qquad (43)$$

or, in other words, the two bodies adhere after the shock, and proceed with a common velocity represented by equation (43).

If the bodies move in opposite directions before the

shock, we have only to change the sign of V' in equations (41).

EXAMPLES.

1. A body weighing 10 lbs., moving at the rate of 5 miles per hour, overtakes a body of 5 lbs., moving at the rate of 3 miles per hour; and their relative coefficient of elasticity is $\frac{1}{3}$; find their velocities after impact. *Ans.* $v = 4\frac{1}{9}$ miles per hour.
$v' = 4\frac{7}{9}$,,

2. If these bodies were perfectly elastic, what would be their velocities after impact? *Ans.* $v = 3\frac{2}{3}$ miles per hour.
$v' = 5\frac{2}{3}$,,

3. A goods train weighing 200 tons, and travelling at 20 miles per hour, in a fog, runs into a passenger train of 50 tons, standing on the same line; find the rate at which the remains of the passenger train will be propelled along the line, supposing the coefficient of shock or elasticity to be $\frac{1}{5}$th. *Ans.* $v' = 19.2$ miles per hour.

Oblique Collision.—Let two bodies, whose masses are M, M', move with velocities V, V', along the lines AA′ and BB′, situated in the same plane, coming into collision at the point O. It is required to find the direction and magnitude of their velocities after the shock has taken place.

Let OX be the line joining the centres, and OY the common tangent of the two spheres; resolving the velocities in the directions OY and OX, we find the components:—

In the direction OX

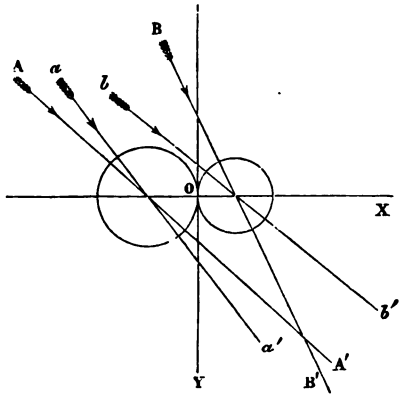
$$V \cos i, \quad V' \cos i',$$

and in the direction OY

$$V \sin i, \quad V' \sin i',$$

i, i' denoting the angles between AA′ and OX, and BB′ and OX respectively.

If, after the shock, aa' and bb' be the directions of the motion, and the angles made by these lines with OX be



r, r', then if v, v' denote the velocities after the shock, their components along OX and OY will be

$$v \cos r, \quad v' \cos r';$$

and

$$v \sin r, \quad v' \sin r'.$$

The components of the original velocity will not be altered in the direction of the common tangent OY, and the components in the direction OX of the line joining the centres will be subject to the rules of direct collision given in equations (41). Hence, we find,

$$V \sin i = v \sin r;$$
$$V' \sin i' = v' \sin r';$$

$$v \cos r = \frac{MV\cos i + M'V'\cos i' - eM'(V\cos i - V'\cos i')}{M + M'}; \quad (44)$$

$$v' \cos r' = \frac{MV\cos i + M'V'\cos i' - eM(V'\cos i' - V\cos i)}{M + M'}.$$

Dividing the first of these equations by the third, and the second by the fourth, we find,

$$\tan r = \frac{(M+M')\,V\sin i}{(M - eM')\,V\cos i + M'(1+e)\,V'\cos i'};$$
$$\tan r' = \frac{(M+M')\,V'\sin i'}{(M' - eM)\,V'\cos i' + M(1+e)\,V\cos i}. \quad (45)$$

These equations express in terms of known quantities the directions of the motion of both bodies after the oblique shock has taken place. To find the velocities after collision, we must add together the squares of the first and third, second and fourth of (44); from which we find,

$$v^2 = V^2\sin^2 i + \left(\frac{(M - eM')\,V\cos i + M'(1+e)\,V'\cos i'}{M+M'}\right)^2;$$
$$v'^2 = V'^2\sin^2 i' + \left(\frac{(M' - eM)\,V'\cos i' + M(1+e)\,V\cos i}{M+M'}\right)^2; \quad (46)$$

Miscellaneous Exercises.

1. If a ball fall from a height of 100 ft., and rebound to a height of 70 ft., what is the coefficient of elasticity? *Ans.* $e = 0.836$.

2. A ball falling from a height of 100 ft. hops four times on the surface of a body, with which its common coefficient of elasticity is $\frac{1}{2}$; find the height of the fourth hop. *Ans.* 0.39 ft.

3. A goods train of 200 tons, moving at 20 miles per hour, meets in a tunnel, on a single line of rails, a passenger train of 50 tons, moving at the rate of 40 miles per hour. If the elasticity of the engines shocking against each other be represented by $\frac{1}{4}$, find the motion of each train after the shock. *Ans.* $v = 5$ miles per hour.
$v' = 20$,,

Or, the goods train will continue its course at the diminished speed of

5 miles per hour, and the passenger train will have its entire motion destroyed, and be forced back at the rate of 20 miles per hour.

4. An imperfectly elastic sphere, whose coefficient of elasticity is $e = \tan 30°$, impinges upon a plane in such a manner that its velocity after impact is to its velocity before impact as sin 45° to radius. Find the angles of incidence and reflexion. (*Wrigley.*) *Ans.* $i = 30°$.
$r = 45°$.

5. Four balls, A, B, C, D, of perfect elasticity, are placed in a straight line. Find the ratio of their masses, so that the quantity of motion in A may be equally divided among the four balls after the shock. (*Wrigley.*) *Ans.* A : B : C : D = 10 : 6 : 3 : 1.

6. A perfectly elastic sphere impinges upon an equal sphere at rest, so that the line joining their centres at the impact makes an angle of 45° with the line of approach of the first sphere; find the angle be-between their paths after the shock. (*Wrigley.*) *Ans.* 90°.

7. Two spheres M and 2M, whose coefficient of elasticity is $\frac{2}{3}$, move with velocities $2V$ and V respectively, and the direction of each motion makes an angle of 30° with the common tangent at the point of impact. Find the directions and velocities of both spheres after the shock.

Ans. The angles made with the common tangent are equal, and each 21° 3′ 6″.
The velocity of M = $2V \times 0.928$.
The velocity of 2M = $V \times 0.928$.

8. Two perfectly elastic spheres, M and M′, meet directly with equal velocities; find the relation between their masses, so that, after collision, one of them may remainat rest. (*Wrigley.*) *Ans.* M = 3M′,
or, M′ = 3M.

9. A weight P, after falling freely through h feet, begins to pull up a heavier body Q, by means of a cord passing over a pulley, as in Atwood's machine. Find the height through which it will lift it. (*Wrigley.*)

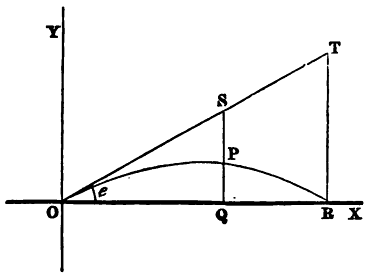
Ans. $\dfrac{P^2 h}{Q^2 - P^2}$.

CHAPTER VII.

ON PROJECTILES.

1. Motion of a heavy Body projected obliquely.—2. Expression for the Direction and Velocity.—3. Time of Flight on a Horizontal Plane.—4. Range on a Horizontal Plane.—5. Greatest Vertical Height over a Horizontal Plane.—6. Time of Flight on an Oblique Ascending Plane.—7. Range on an Oblique Ascending Plane.—8. Time and Range on an Oblique Descending Plane.—9. The Velocity and Angle of Projection of a Trajectory, two Points in which are given.—10. Velocity of Discharge.

1. Motion of a heavy Body projected obliquely.—If a heavy body be projected vertically, either upwards or downwards, its path will evidently be a right line. If it be projected in an oblique direction, it is a matter of experience, that its path will be a curved line, concave to the horizontal plane. The nature of this path, or *trajectory*, as it is generally called, and the general circumstances of the motion of the projectile, may be investigated as follows:—



Let OX be a horizontal, and OY a vertical line drawn through O, the point of projection; let OT be the direction in which the body is projected; and OPR its path or trajectory. If the body be projected with a velocity V, and if we suppose it to move uninfluenced by the force of gravity, it will be found at the end of the time t in the line OT at a distance OS = Vt, but as it is drawn

downwards by the force of gravity during the whole of its motion, its actual position P, at the end of the time t, will be found by measuring SP vertically, downwards, and equal to $\frac{1}{2}gt^2$. If now we denote by e the angle of projection XOT, by x the horizontal distance OQ, and by y the vertical height PQ, the position of the projectile at any time t may be expressed by its co-ordinates x and y as follows:—

$$x = Vt \cos e$$

$$y = x \tan e - \frac{gt^2}{2}. \qquad (47)$$

If t be eliminated from these equations, we obtain the following relation between x and y—

$$y = x \tan e - \frac{gx^2}{2 V^2 \cos^2 e}.$$

Let h be the height due to the velocity V, then $V^2 = 2gh$; substituting this value in the last equation, we obtain—

$$y = x \tan e - \frac{x^2}{4h \cos^2 e}. \qquad (48)$$

This is the equation of a parabola passing through the point of projection, and having its axis perpendicular to the horizon.

2. **Expression for the Direction and Velocity.**— If the velocity of projection V be resolved parallel and perpendicular to the horizon, the horizontal component $V \cos e$, being at right angles to the direction of gravity, will be altogether uninfluenced by that force, and will, therefore, remain constant during the whole time of *flight;* the vertical component $V \sin e$, being directly

opposed to gravity, will at the end of the time t be diminished by gt, so that the vertical component of the velocity of the projectile at any instant will be expressed by $V \sin e - gt$. If, therefore, the velocity of the projectile at any time be expressed by v, and if ϕ be the angle which the direction of its motion makes with the horizontal line, we have

$$v \sin \phi = V \sin e - gt;$$
$$v \sin \phi = V \cos e.$$

By dividing one of these equations by the other,

$$\tan \phi = \tan e - \frac{gt}{V \cos e}; \qquad (49)$$

by adding their squares together—

$$v^2 = V^2 - 2\,Vgt \sin e + g^2t^2.$$

Since the velocity of projection in the vertical direction is $V \sin e$, the space y described in that direction is expressed by

$$y = Vt \sin e - \frac{gt^2}{2};$$

multiply this equation by $2g$, and add it to that last obtained, and we have

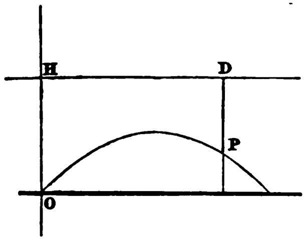
$$v^2 + 2gy = V^2;$$

but since $V^2 = 2gh$,

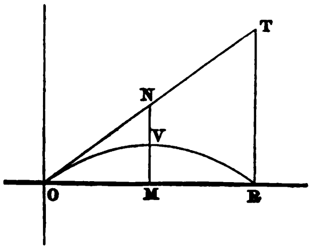
$$v^2 = 2g\,(h - y), \qquad (50)$$

the velocity v is, therefore, due to the height $h - y$, from which we have the following theorem :—

If through a point H *vertically over the point of projection, and at a distance from it* OH *equal to the height due to the velocity of projection, a line be drawn parallel to the horizon, the velocity at any point of the trajectory will be equal to that acquired by falling from* D, *the point of the parallel vertically over* P.



3. **Time of Flight on a Horizontal Plane.**—If T be the time of flight, it is evident that OT = VT and TR = $\frac{1}{2}gT^2$; but as OTR is a right-angled triangle, TR = OT sin e, and therefore



$$\tfrac{1}{2}gT^2 = VT \sin e,$$

from which

$$T = \frac{2V \sin e}{g}. \tag{51}$$

4. **Range on a Horizontal Plane.**—Let the range OR be denoted by R; it follows from the right-angled triangle OTR that R = OT cos e = VT cos e; substituting for T in this equation its value from (51), we obtain

$$R = \frac{2V^2}{g} \sin e \cos e,$$

or, as $V^2 = 2gh$, and $2 \sin e \cos e = \sin 2e$,

$$R = 2h \sin 2e. \tag{52}$$

As the greatest value of $\sin 2e$ corresponds to $2e = 90°$, it follows—

That for a given velocity, the maximum range on a horizontal plane corresponds to an elevation of $45°$.

From this it follows, that for a maximum range OR = RT, or

$$R = 16\ T^2;$$

and therefore

$$T = \tfrac{1}{4}\ \sqrt{\text{maximum range}},$$

from which we have the following rule:—

The time of flight in seconds corresponding to the maximum range is equal to one quarter of the square root of the maximum range in feet.

If the velocity of projection and the horizontal range be given, the elevation may be found from the equation

$$\sin 2e = \frac{R}{2h}.$$

But as the sines of supplemental angles are equal, it follows that for a given range there are two elevations e, e', such that

$$2e + 2e' = 180°,$$

or

$$e + e' = 90°.$$

From this it appears—

That the same point in a horizontal plane passing through the point of projection may be reached by two projectiles whose angles of elevation are complemental.

5. **Greatest Vertical Height over a Horizontal Plane.**—It is evident that the projectile will reach its greatest vertical height at the middle of its flight; if, therefore, we substitute in equation (48) for x, the value of half the range, viz., $h \sin 2e$, the resulting value of y will be the greatest height to which the projectile will ascend over the horizontal line—

$$\text{Greatest height} = h \sin^2 e. \tag{53}$$

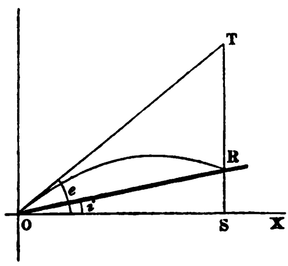
EXAMPLES.

1. To what distance, measured on a horizontal plane, will a shell be projected, which is discharged with a velocity of 520 ft. per second, and at an elevation of 36°? What is the time of flight?
Ans. Range = 8036 feet.
Time of flight = 19.1 seconds.

2. For the same velocity calculate the maximum range and time of flight.
Ans. Range = 8450 feet.
Time of flight = 22.9 seconds.

3. At a distance of a quarter of a mile from the bottom of a cliff, which is 120 feet high, a shot is to be fired which shall just clear the cliff, and pass over it horizontally; calculate the angle and velocity of projection.
Ans. Angle of projection = 10° 18′.
Velocity of projection = 490 ft. per sec.

6. **Time of Flight on an Oblique Ascending Plane.**—If the plane OR, on which the range is measured, lie above the horizontal line OX, making with it an angle ROX = i; let the angle of elevation XOT = e, then in the oblique-angled triangle ORT, the angle ROT = $e-i$, and the sides OT, TR, and RO are equal to VT, $\frac{1}{2}g\,T^2$ and R respectively; from this, and by the rule of sines, we may at once es-



tablish, as in the former case, the relations between the angle of projection, the velocity of projection, the range and time of flight: thus, since

TR : OT : : sin ROT : sin ORT,

we have

$$\frac{gT^2}{2} : VT :: \sin(e - i) : \cos i,$$

and therefore

$$T = \frac{2V}{g} \frac{\sin(e - i)}{\cos i}. \qquad (54)$$

7. **Range on an Oblique Ascending Plane.**—From the same triangle

OR : OT : : sin OTR : sin ORT,

or

$$R : VT :: \cos e : \cos i;$$

therefore,

$$R = VT \frac{\cos e}{\cos i};$$

or, substituting for T its value from equation (54)

$$R = \frac{2V^2}{g} \frac{\cos e \sin(e - i)}{\cos^2 i},$$

and as $V^2 = 2gh$, we obtain

$$R = \frac{4h \cos e \sin(e - i)}{\cos^2 i}. \qquad (55)$$

If in the last equation we substitute for $2 \cos e \sin(e-i)$

its value, $\sin(2e - i) - \sin i$, and solve for $\sin(2e - i)$, we obtain

$$\sin(2e - i) = \frac{R\cos^2 i}{2h} + \sin i. \qquad (56)$$

From this equation it follows that the greatest range on an oblique ascending plane corresponds to an elevation e such that $2e - i = 90$. If this value of e be denoted by ϵ,

$$2\epsilon = 90 + i;$$

and therefore, for a given velocity of projection,

The direction of elevation which corresponds to the maximum range on an oblique plane bisects the angle between the plane and the vertical line passing through the point of projection.

From this it follows that the maximum range OR = RT, and therefore that $16\,T^2 = R$, and therefore that the rule for finding the time of flight given in page 149 is also true for the maximum range measured on oblique planes.

If the velocity of projection and the range be given, the elevation may be found from equation (56); but as the sines of supplemental angles are equal, it follows that for each range there are two elevations e and e', such that

$$2e - i + 2e' - i = 180,$$

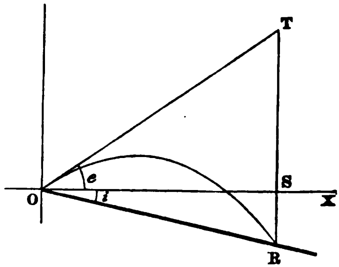
or

$$e + e' = 90 + i = 2\epsilon;$$

from which it appears,

That the same point on an oblique plane passing through the point of projection may be reached by two projectiles whose lines of elevation make equal angles on each side of the line of elevation corresponding to the maximum range.

8. **Time and Range on an Oblique Descending Plane.**—If the plane OR on which the range is taken fall *below* the horizontal line OX, making with it an angle i, the expressions for the time of flight and the range may be similarly obtained; they are as follows:—



$$T = \frac{2V}{g}\,\frac{\sin(e+i)}{\cos i}. \qquad (57)$$

$$R = 4h\,\frac{\cos e \sin(e+i)}{\cos^2 i}. \qquad (58)$$

Substituting in this for $2 \cos e \sin (e+i)$ its value, $\sin (2e+i) + \sin i$, we obtain

$$\sin(2e+i) = \frac{R\cos^2 i}{2h} - \sin i.$$

If ϵ be the elevation corresponding to the maximum range,

$$2\epsilon = 90 - i,$$

and if e and e' be the two elevations which correspond to the same range,

$$e + e' = 90 - i = 2\epsilon.$$

It is evident, therefore, that the theorems which have been established in the last article apply also to the case of oblique descending planes.

EXAMPLES.

1. A ball is fired up a hill whose inclination is 15°; the inclination of the piece is 45°, and the velocity of projection 500 feet per second; calculate the time of flight before it strikes the hill, and the distance of the place where it falls from the point of projection.

Ans. $R = 1.121$ miles.
$T = 16.17$ seconds.

2. On a descending plane, whose inclination is 12°, a ball fired from the top hits the plane at a distance of two miles and a half; the elevation of the piece is 42°; calculate the velocity of projection.

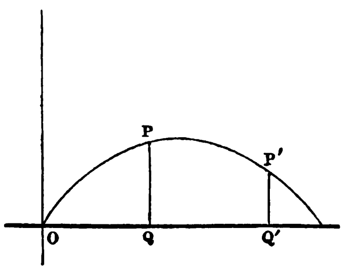
Ans. $V = 579.74$ ft. per sec.

3. A ball is fired down a descending plane, whose fall is 10°; the range is found to be one mile, and the time of flight 20 seconds; calculate the elevation. *Ans.* 46° 28′.

4. Calculate the maximum range and time of flight on a descending plane, the inclination of which is 15°, the velocity of projection being 1000 ft. per sec. *Ans.* Maximum range = 7.98 miles.
Time of flight = 51.34 seconds.

5. With what velocity does the ball strike the plane in the last question? *Ans.* Velocity = 1303 feet.

9. The Velocity and Angle of Projection of a Trajectory, two Points in which are given.—If P and P′ be two points on the trajectory, the velocity and angle of projection may be investigated as follows:—Let OQ and QP, the co-ordinates of P, be denoted by p and q; OQ′ and Q′P′ the co-ordinates of P′, by p' and q'; then, since P and P′ are points on the trajectory, it follows from equation (48), if e be the



angle of elevation and h the height due to the velocity of projection, that

$$q = p \tan e - \frac{p^2}{4h \cos^2 e},$$

and

$$q' = p' \tan e - \frac{p'^2}{4h \cos^2 e}.$$

Eliminating h from these equations by cross multiplication, we obtain for the elevation

$$\tan e = \frac{qp'^2 - q'p^2}{pp'(p' - p)}. \tag{59}$$

Solving for h from the first equation, we obtain

$$h = \frac{p^2 \sec^2 e}{4(p \tan e - q)}, \tag{60}$$

from which the velocity of projection may be calculated.

10. **Velocity of Discharge.** — The velocity with which a ball or shell is fired from the mouth of a gun must depend (the quality of the powder being supposed uniform), on the weight of the ball, the quantity of powder, and the length of the piece. As the latter has been shown by experience to have but little influence, we may assume that the velocity of discharge depends only on the weight of the ball and the weight of the charge. The formula by which it is expressed is derived partly from theoretical considerations, and partly from experiment. If we suppose that the powder explodes just in proportion as the ball advances in the bore of the gun, it is evident that it will be pressed forwards by a constant force; and, therefore, that its velocity will be uniformly accelerated until it reaches the mouth. Let F represent this constant force, m the mass of the ball, and f the

velocity produced in one second, we have, by equation (2),

$$f = \frac{F}{m}.$$

If V be the velocity at the mouth, and l the length of the piece, we have, by equation (10),

$$V^2 = \frac{2Fl}{m}.$$

Now m is proportional to the weight of the ball W, and F may be assumed to be in proportion to the weight of the powder P; from this it follows that

$$V^2 = \frac{kP}{W},$$

in which equation k is a constant coefficient, which may be determined experimentally. As the result of numerous experiments conducted on Woolwich Warren by Dr. Hutton, the following formula may be assumed as a correct expression for the velocity of balls and shells expressed in feet per second, fired with moderate charges of powder, from the pieces of ordnance commonly used for military purposes,—

$$V = 1600 \sqrt{\frac{3P}{W}}. \tag{61}$$

EXAMPLES.

1. The weight of a 13-inch shell is 196 lbs., and the charge of powder 9 lbs.; calculate the velocity of discharge.
Ans. 594 ft. per sec.

2. An 8-inch shell weighs 48 lbs., and is fired with 2 lbs. of powder; find the velocity of discharge. *Ans.* 565 ft. per sec.

3. A 32 lb. shot is fired with a charge of 6 lbs. of powder; find the velocity of discharge. *Ans.* 1200 ft. per sec.

Miscellaneous Examples.

1. If a 13-inch shell be fired with a charge of 9 lbs., what is the maximum horizontal range and time of flight?

Ans. Range = 3675 yards.
Time = $26\frac{1}{4}$ seconds.

2. How much powder will throw a 13-inch shell 4000 ft. on an inclined plane which ascends 10° 40′, the elevation of the mortar being 35°. *Ans.* Charge = 4.67 lbs.

3. A 32 lb. shot is fired with 3 lbs. of powder; calculate the elevation of the piece in order that it shall strike an object 250 ft. above the horizontal line, and at a distance of three-quarters of a mile.

From equation (48) we find the following expression for the elevation, which corresponds to a trajectory passing through a point y, x—

$$\text{Tan } e = \frac{2h \pm \sqrt{4h(h-y) - x^2}}{x}.$$

By means of this expression, and the value of $h = 11250$, found by equation (61) we obtain *Ans.* Elevation = 8° 42′.
or = 84° 54′.

4. A projectile is discharged in a horizontal direction, with a velocity of 450 ft. per sec., from the summit of a conical hill, the vertical angle of which is 120°: at what distance down the side of the hill will the projectile fall, and what will be the time of flight?

Ans. Distance = 2812.5 yards.
Time of flight = 16.23 seconds.

5. A gun is placed at a distance of 500 feet from the base of a cliff which is 200 feet high; on the edge of the cliff there is built the wall of a castle 60 feet high; find the elevation of the gun, and the velocity of discharge, in order that the ball may graze the top of the castle wall, and fall 120 feet inside it.

Ans. Elevation = 53° 19′.
Velocity = 165 ft. per sec.

6. A ball 13 lbs. weight is fired from a gun on the summit of a mountain, with a charge of $6\frac{1}{2}$ oz. of powder, so as just to strike the bottom of the mountain with double the velocity of projection; find the height of the mountain. *Ans.* 2.11 miles.

7. A piece of ordnance, lately under proof at Woolwich, burst when 50 yards from a wall 14 feet high, and a fragment of it, originally in contact with the ground, after grazing the wall, fell 6 feet beyond it on the opposite side. Find how high it rose in the air.

Ans. 94 ft.

8. Two projectiles, fired with velocities due to the heights h_1, h_2, at elevations e_1, e_2, strike the same point on the side of the hill on which the gun is placed; find the slope of the hill. (*Newth.*)

$$\textit{Ans. } \tan i = \frac{1}{2}\,\frac{h_1 \sin 2e_1 - h_2 \sin 2e_2}{h_1 \cos^2 e_1 - h_2 \cos^2 e_2}.$$

ERRATUM.

For $\frac{8-\pi}{5}$, and $\frac{6-\pi}{5}$, in the answer to Example 27, page 38, *read* $\frac{8-\pi}{6-\pi}$, and 1.